Integrated Green function for charged particle moving along bending orbit

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Overview

Introduction

1. Integrated Green function. K.Ohmi used it to obtain Green function for flat beam collision PRE62, 7287 (2000). J. Qiang called it Integrated Green Function. (K. Yokoya has used it since 1980's.)
2. The integration is used to regularize the potential behavior $G = \log|\boldsymbol{x} - \boldsymbol{x}'|$. It was essential to realize a correct beam-beam force for very flat beam with aspect ratio $< 1/100$.
3. Electro-magnetic field near moving single electron is calculated under. free boundary condition. Longitudinal wake in 2d was done by C. Huang et al. PR ST-AB 16, 010701 (2013).
4. Integrated Green Function is used to regularize to sigular behavior of high frequency component of EM field. Similar works were done by R. D. Ryne et al.
5. Integrated Green Function, which is equivalent to Longitudinal and Transverse (mono-pole) wake field, were obtained.

Lienard-Wiechert potential

Electro-magnetic potential of moving elelctrons, Lienard-Wiechert potential.

$$\phi(\boldsymbol{x},t)=\frac{1}{4\pi\varepsilon_0}\frac{e}{R(1-\boldsymbol{n}\cdot\boldsymbol{\beta}')} \qquad \boldsymbol{A}(\boldsymbol{x},t)=\frac{\mu_0}{4\pi}\frac{e\boldsymbol{v}'}{R(1-\boldsymbol{n}\cdot\boldsymbol{\beta}')} \tag{1}$$

Field is evaluated at $\boldsymbol{x}$
Electron position is $\boldsymbol{x}'$

$$\boldsymbol{R}=\boldsymbol{x}-\boldsymbol{x}' \qquad \boldsymbol{n}=\frac{\boldsymbol{R}}{R} \qquad \boldsymbol{v}'=d\boldsymbol{x}'/dt' \qquad \boldsymbol{\beta}'=\boldsymbol{v}'/c \tag{2}$$

Space time position of moving electron is $(\boldsymbol{x}', t')$ and field is observed at $(\boldsymbol{x}, t)$

$$t=t'+\frac{R}{c} \tag{3}$$

Electro-magnetic field of moving electron

Electric filed given by Lienard-Wiechert potential.

$$\boldsymbol{E} = \frac{e}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}-\boldsymbol{\beta}'}{\gamma^2\kappa^3 R^2} + \frac{\boldsymbol{n}\times((\boldsymbol{n}-\boldsymbol{\beta}')\times\boldsymbol{\alpha}')}{\kappa^3 R}\right] \tag{4}$$

$$= \frac{e}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}-\boldsymbol{\beta}'}{\gamma^2\kappa^3 R^2} + \frac{\boldsymbol{n}\cdot\boldsymbol{\alpha}'(\boldsymbol{n}-\boldsymbol{\beta}') - (1-\boldsymbol{n}\cdot\boldsymbol{\beta}')\boldsymbol{\alpha}'}{\kappa^3 R}\right]$$

$$\equiv \frac{e}{4\pi\varepsilon_0}(\boldsymbol{E}^{(c)} + \boldsymbol{E}^{(r)}) \qquad \text{Coulomb and radiation terms}$$

$$\kappa \equiv 1 - \boldsymbol{n}\cdot\boldsymbol{\beta}' \qquad \boldsymbol{\alpha}' \equiv d\boldsymbol{\beta}'/dct'$$

Magnetic field

$$\boldsymbol{B} = \frac{1}{c}\boldsymbol{n}\times\boldsymbol{E} \tag{5}$$

Lorentz force

$$\boldsymbol{F} = e(\boldsymbol{E} + c\boldsymbol{\beta}\times\boldsymbol{B}) = e\left[\boldsymbol{E} + \boldsymbol{\beta}\times(\boldsymbol{n}\times\boldsymbol{E})\right]$$

$$= e\left[(1-\boldsymbol{n}\cdot\boldsymbol{\beta})\boldsymbol{E} + (\boldsymbol{\beta}\cdot\boldsymbol{E})\boldsymbol{n}\right] \tag{6}$$

Electro-magnetic field for an electron with given trajectory

Electro-magnetic field is calculated on mesh points near a source electron.

1. Area near a source electron is devided to $n_x \times n_y \times n_z$ mesh with the size $\Delta x \times \Delta y \times \Delta z$, where $\Delta x, y, z \sim \sigma_{x,y,z}/10$ depending on studied physics.
2. The field has very high frequency component, $\lambda_c = 4\pi\rho/3\gamma^3 \ll \Delta x, y, z$ in general.
3. Integrated Green Function, which Lienard-Wiechert field is integrated with constant weight in the area $\Delta x \Delta y \Delta z$, is introduced.

$$\boldsymbol{F}(\vec{x}_{ijk}, s; 0, 0) = \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} \int_{y_j-\Delta y/2}^{y_j+\Delta y/2} \int_{z_k-\Delta z/2}^{z_k+\Delta z/2} \boldsymbol{F}(\vec{x}, s; 0, 0) d\vec{x}. \qquad (7)$$

Accelerator coordinates and Space time relation

Evaluate electro-magnetic field and source particle motion are expressed by the space time $\boldsymbol{x} = (x, y, s)$ as function of t.
The space time relation of L-W potential

$$t = t' + \frac{R(x, y, s - s')}{c} \qquad t' = t'(s') \tag{8}$$

Accelerator coordinate $\vec{x}(s) = (x, y, z)$ as function of s, where $z = \beta c(t' - t)$ is arrival time advance. L-W field is induced by an electron $\vec{x}' = (x', y', s')$ at $z'(s)$, with the assumption $z'(s') = z'(s)$ (next page).
The space time relation

$$s = s' + \beta R(x, y, s - s') + z. \tag{9}$$

The source particle move along reference robit, choose $z'(s) = z'(s') = 0$.

Green function and integrated Green function

Green function ≈ Field given by single point charge, L-W field.
Force for a charge distribution $\Psi(\vec{x}'\vec{p}')$, $\psi(\vec{x}') = \int \Psi d\vec{p}'$

$$\bar{\boldsymbol{F}}(\vec{x},\vec{p},s) = \int \boldsymbol{F}(\vec{x},\vec{p},s;\vec{x}',\vec{p}',s')\Psi(\vec{x}',\vec{p}';s')d\vec{x}'d\vec{p}' \tag{10}$$

First assumption: particles move the same trajectory. $\beta_{xy} > s - s'$

$$\bar{\boldsymbol{F}}(\vec{x};s) \approx \int \boldsymbol{F}(\vec{x},s;\vec{x}')\psi(\vec{x}',s')d\vec{x}' \tag{11}$$

Integrated Green function, with second assumtion that the distribution does not change in the motion of s' to s. s' is function of $s'(\vec{x}-\vec{x}',s)$.

$$\bar{\boldsymbol{F}}(\vec{x}_{ijk};s) \approx \sum_{lmn} \boldsymbol{F}(\vec{x}_{ijk},s;\vec{x}'_{lmn},s')\psi(\vec{x}'_{lmn},s) \tag{12}$$

$$\boldsymbol{F}(\vec{x}_{ijk},s;\vec{x}'_{lmn},s') = \boldsymbol{F}(\vec{x}_{ijk}-\vec{x}'_{lmn},s) \tag{13}$$

Integration of Green function

1. Green function/LW field contains high frequency component.
2. The detailed behavior has to be considered to integrate LW field.
3. Particle distribution is assumed uniform in the integration area. The integration is regarded as a cutoff the high frequency component.

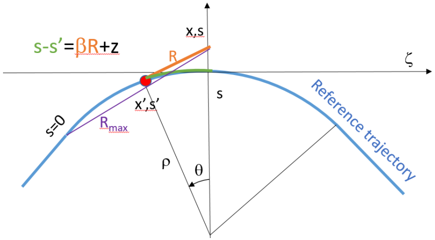

Figure: Definition of coordinates. x, y

Motion of Source electron

LW field near the source particle moving in bending magnet is calculated. Beam orbit is expressed as function of s' by

$$x'(s') = \rho\,(\cos\theta - 1) \qquad \zeta'(s') = -\rho\sin\theta \tag{14}$$

$$\beta'_x(s') = \frac{dx'}{dct'} = \beta\sin\theta \qquad \beta'_\zeta(s') = \beta\cos\theta \tag{15}$$

$$\alpha'_x(s') = \frac{d\beta'_x}{dct'} = -\frac{\beta^2}{\rho}\cos\theta \qquad \alpha'_\zeta(s') = \frac{\beta^2}{\rho}\sin\theta \tag{16}$$

$$s' = \beta ct' \qquad s = \beta ct \qquad \theta = \frac{s - s'}{\rho}$$

$$R_{max,ij} = \sqrt{\zeta'(0)^2 + (x_i - x'(0))^2 + y_i^2} \quad s < \beta R_{max,i,j} + z_k \tag{17}$$

Relation between retarded time $s' = \beta ct'$ and z

The space time relation

$$s = s' + \beta R(x, y, s - s') + z. \tag{18}$$

Relation between z and s' at (x, y, s) is obtained.

1. For given s', calculate z. Simple substitution.
2. For given z, calculate s'. Root finding for example using Newton-Raphson method.

$$f(s') = s' - s + \beta R(x, y, s - s') + z, \qquad \frac{df}{ds'} = 1 - \frac{(x - x')\beta'_x - \zeta'\beta'_\zeta}{R}. \tag{19}$$

N-R iteration

$$s'_{i+1} = s'_i - \frac{f(s'_i)}{df/ds'|_{s'_i}}$$

Field as function of $x_\rho = x/\rho, y_\rho = y/\rho, \theta = (s - s')/\rho$

Space-time relation uusing $\theta = (s - s')/\rho$

$$z_\rho = \theta - \beta R_\rho \qquad R_\rho \equiv \frac{R}{\rho} = \sqrt{4\left(1 + x_\rho\right)\sin^2\frac{\theta}{2} + x_\rho^2 + y_\rho^2} \tag{20}$$

$$\boldsymbol{n} = \frac{1}{R_\rho}\left(x_\rho + 2\sin^2\frac{\theta}{2}, y_\rho, \sin\theta\right)$$

$\kappa = -dz/ds' = dz_\rho/d\theta$

$$\kappa = 1 - \boldsymbol{n}\cdot\boldsymbol{\beta} = 1 - \frac{\beta(1 + x_\rho)\sin\theta}{R_\rho} = \tag{21}$$

$$\frac{4\sin^4\frac{\theta}{2} + \frac{1}{\gamma^2}\sin^2\theta + (4\sin^2\frac{\theta}{2} - 2\beta^2\sin^2\theta)x_\rho + (1 - \beta^2\sin\theta)x_\rho^2 + y_\rho^2}{R_\rho(R_\rho + \beta(1 + x_\rho)\sin\theta)}$$

1st and 2nd expressions are usefull for $\theta < 0$ and $\theta > 0$, respectively.

Radiation Field/Force

$$E_s^{(r)} = \frac{\beta^2}{\rho^2 R_\rho^3 \kappa^3} \left[2\sin^2\frac{\theta}{2}(R_\rho\beta - \sin\theta) + \{R_\rho\beta - \sin\theta(2-\cos\theta)\}\, x_\rho - (x_\rho^2 + y_\rho^2)\sin\theta \right] \tag{22}$$

$$F_x^{(r)} = \frac{\beta^2}{\rho^2 R_\rho^3 \kappa^3} \left[\sin^2\theta + 4\beta^2\sin^2\frac{\theta}{2} - 2R_\rho\beta\sin\theta + \left(\sin^2\theta + 4\beta^2\sin^2\frac{\theta}{2} - R_\rho\beta\sin\theta \right) x_\rho + \beta^2 x_\rho^2 + y_\rho^2\cos\theta \right] \tag{23}$$

$$F_y^{(r)} = \frac{\beta^2}{\rho^2 R_\rho^3 \kappa^3} \left[2\sin^2\frac{\theta}{2}(1+\beta^2) - R_\rho\beta\sin\theta + \left(2\sin^2\frac{\theta}{2} - \frac{1}{\gamma^2} \right) \right] y_\rho \tag{24}$$

Numerical cancellation sometimes occurs. Check using quadratic precision numbers.

Coulomb Field/Force

$$F_x^{(c)} = \frac{1}{\gamma^2\kappa^3\rho^2 R_\rho^3}\left[2(1+\beta^2)\sin^2\frac{\theta}{2} - \beta\sin\theta R_\rho + \left(\frac{1}{\gamma^2} + 2\beta^2\sin^2\frac{\theta}{2}\right)x_\rho\right] \tag{25}$$

$$F_y^{(c)} = \frac{1}{\gamma^2\kappa^3\rho^2 R_\rho^3}\left(\frac{1}{\gamma^2} + 2\beta^2\sin^2\frac{\theta}{2}\right)y_\rho \tag{26}$$

$$F_s^{(c)} = \frac{1}{\gamma^2\kappa^3\rho^2 R_\rho^3}\left(\sin\theta - \beta^2\cos\theta R_\rho\right) \tag{27}$$

Radiation Field/Force near source electron

It is wellknown the behavior for $x_\rho = y_\rho = 0$ and $\theta \to 0$.

$$R_\rho(0,0,\theta) = 2\sin\left|\frac{\theta}{2}\right| \qquad \kappa(0,0,\theta) = 1 - \beta\,\mathrm{sgn}(\theta)\cos(\theta/2) \tag{28}$$

$$\begin{aligned} E_s(0,0,\theta) &= \frac{\beta^2}{\rho^2}\frac{\beta - \mathrm{sgn}(\theta)\cos(\theta/2)}{2\left(1-\beta\,\mathrm{sgn}(\theta)\cos(\theta/2)\right)^3} \\ &\approx -\frac{\beta^2(1+\beta)^2\gamma^4}{2\rho^2}\left[1 - \frac{\gamma^2(1+\beta)\theta^2}{8} + O(\theta^4)\right] \qquad \theta \to +0 \end{aligned} \tag{29}$$

$E_s(0,0,+0) \approx -2\gamma^4/\rho^2$ and $E_s \approx 0$ at $\theta \approx 2/\gamma$; $z \approx \rho/\gamma^3$. This behavior is realized by the characteristic wave length $\lambda_c = 4\pi\rho/(3\gamma^3)$.
Discontinuity for sgn function gives small number ($\approx 1/(8\rho^2)$) for $\theta < 0$.

Studies for SACLA $E = 8$ GeV, $\rho = 46.4$ m, $\lambda_c = 0.05$ nm

Emittance growth has been observed at branch line in SACLA. Effect of transverse CSR force is motivation of this study.

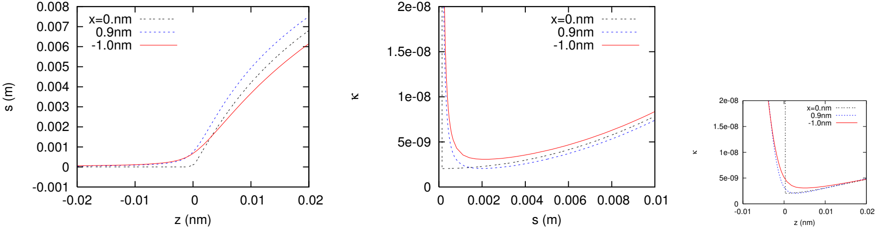

Figure: $s'(z), \kappa(s)$.

- $s'(z)$, κ. Field from upstream contributes.
- For positive z, field from downstream is dominant.
- Behavior of $z \approx 0$ for $x = 0$ is nondifferentiable.
- The behavior is gentle for finite x.

Lorentz force near single particle (radiation part)

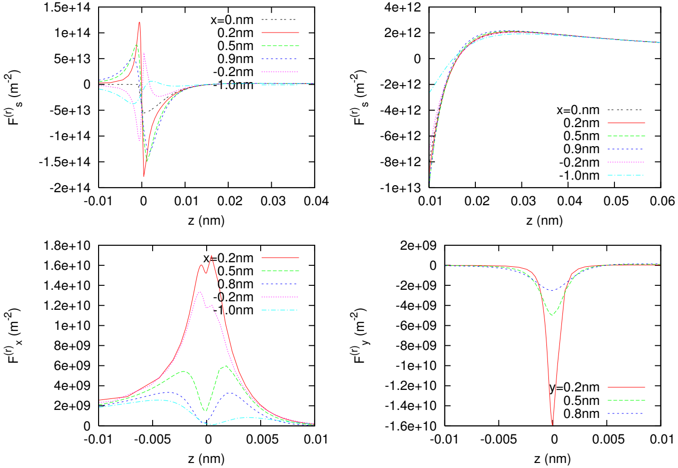

Figure: Lorentz force near a point charge. (top) $F_s^{(r)}$, (bottom) $F_x^{(r)}$ and $F_y^{(r)}$. $\lambda_c = 0.05$ nm, $E_s(0) = -2\gamma^4/\rho^2 = -5.58 \times 10^{13}$ m^{-2} (top left plot x=0).

Convergence of the Integrated Green Function

Integrated Green Function for $\Delta x = \Delta y = 5\ \mu$m, $\Delta z = 0.1\ \mu$m is calculated $\sim 0.1 \times (\sigma_x, \sigma_y, \sigma_z)$.
Integration inside a mesh is performed numerically.
The integration step, $dz/\lambda_c < 1$, $ds \sim 2\gamma^2 dz$, $dx, dy \sim 2\gamma dz$

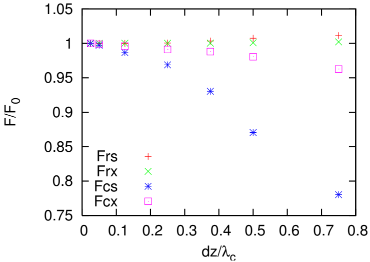

Figure: Convergence of Integrated Green function for integration step in nearest mesh $x = y = z = 0$.

Integrated Longitudinal force (radiation part)

F_s is smaller for $x \sim 1\ \mu$m after integration, while it is larger for small $x \sim 1$ nm in page 17.

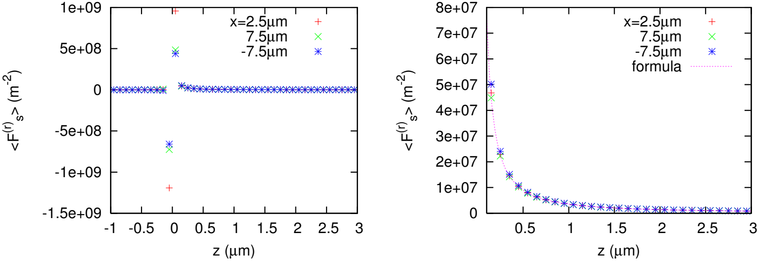

Figure: Longitudinal wake force. Right plot is for front part.

$$\text{Formula} \qquad F_s(z) = \frac{\gamma^4}{2^{1/3}\rho^2}\left(\frac{3\gamma^3}{2\rho}z\right)^{-4/3} \qquad (30)$$

$F_s > 0$ for $z > 0$ means acceleration of particles arriving earlier (leading particles).

Integrated horizontal force (radiation part)

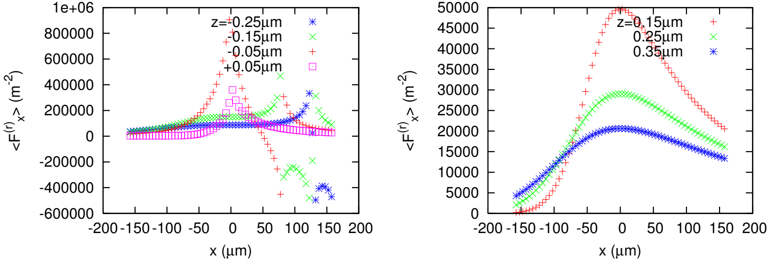

Figure: .Integrated horizontal force. Left and right plots are for $z < 0$ and $z > 0$.

Singular behaviors are seen in positive x for negative z (traling particles).

Field enhancement of tangential direction

Small $\kappa = 1 - \boldsymbol{n} \cdot \boldsymbol{\beta}'$ for tangential direction enhances LW field.

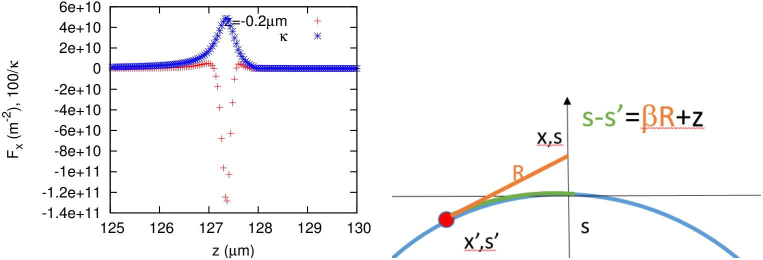

Figure: L-W field and κ near tangential position. $z < 0$ for $\beta R > s - s'$ at large x.

Integrated vertical force (radiation part)

Weak vertical field $F_y^{(r)} \sim F_x^{(r)}/100$.
Slope changes for $z < 0$ (focusing) and $z > 0$ (defocusing).

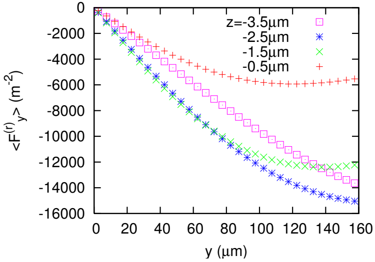

Figure: Integrated vertical force. Left and right plots are for $z < 0$ and $z > 0$.

Integrated Longitudinal force of Coulomb part

$$\boldsymbol{F}^{(c)} = \frac{\boldsymbol{n} - \boldsymbol{\beta}'}{\gamma^2 \kappa^3 R^2} \tag{31}$$

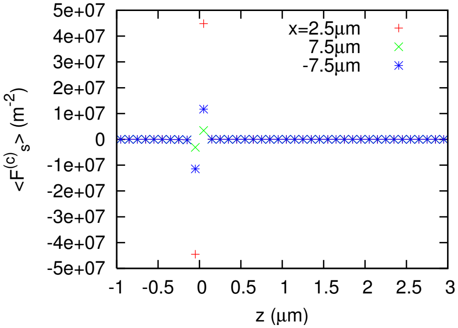

Figure: Integrated Longitudinal force of Coulomb part.

Integrated Transverse Force of Coulomb part

Coulomb force is smaller than radiation force, $F^{(c)} \sim F^{(r)}/100$.

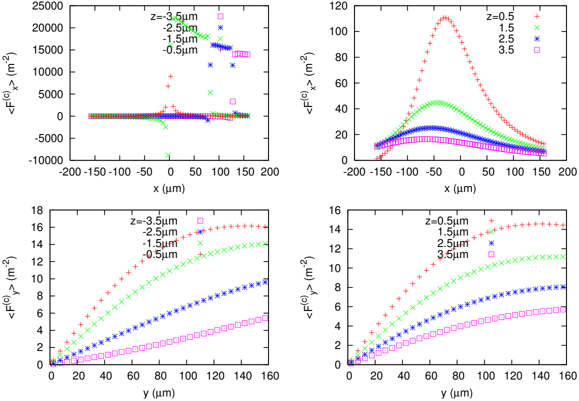

Figure: Integrated Transverse Green function of Coulomb force. Top left $F_x^{(c)}(x, y = 0, z < 0)$, top right $F_x^{(c)}(x, y = 0, z > 0)$, bottom left $F_y^{(c)}(x = 0, y, z < 0)$, bottom right $F_y^{(c)}(x = 0, y, z > 0)$

Integrated Longitudinal Force (E=8.0, 1.0, 0.2 GeV)

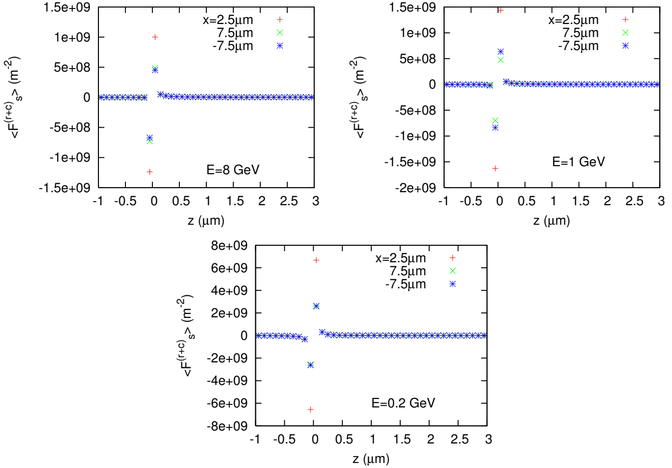

Figure: Longitudinal Integrated Green function. Sum of the radiation and Coulomb forces, $F_s^{(r)} + F_s^{(c)}$.

Integrated Transverse Force (8 GeV)

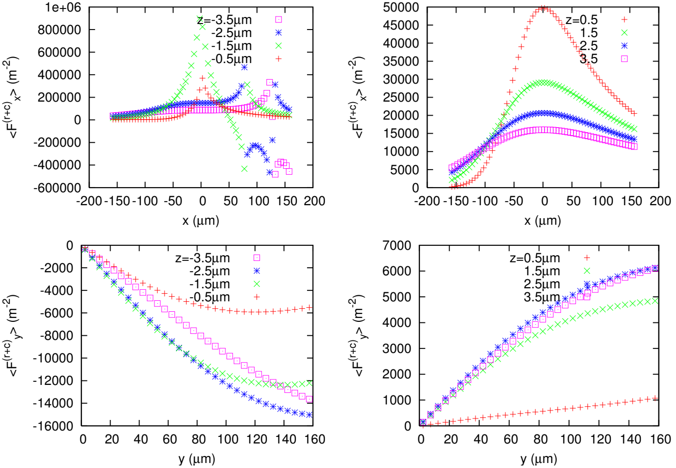

Figure: Transverse Integrated Green function at E=8 GeV. Sum of the radiation and Coulomb forces, $F^{(r)}_{x/y} + F^{(c)}_{x/y}$.

Integrated Transverse Force (1 GeV)

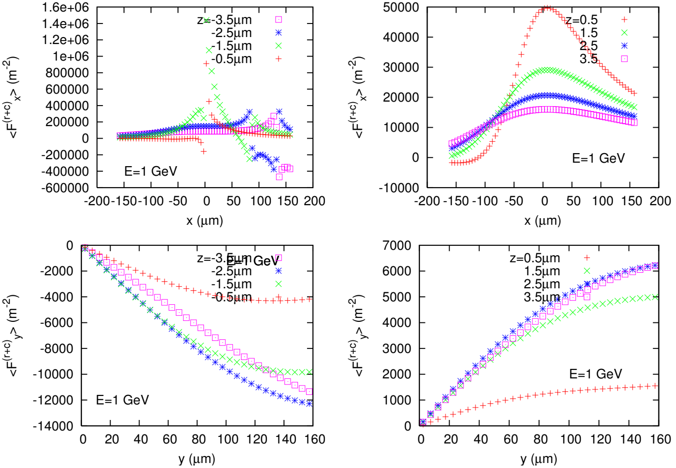

Figure: Transverse Integrated Green function at E=1 GeV. Sum of the radiation and Coulomb forces, $F^{(r)}_{x/y} + F^{(c)}_{x/y}$.

Integrated Transverse Force (0.2 GeV)

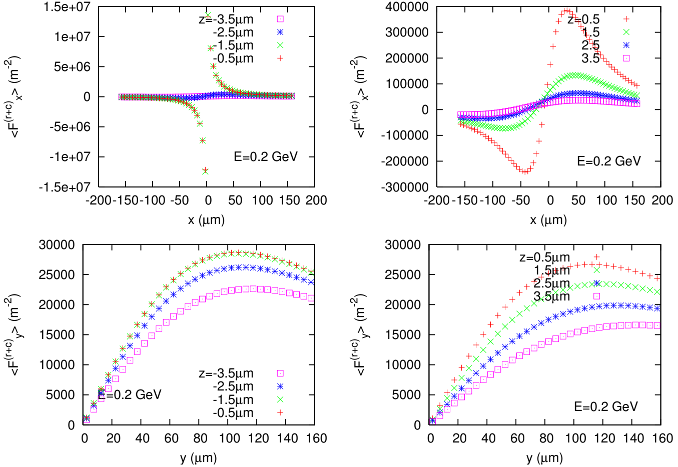

Figure: Transverse Integrated Green function at E=0.2 GeV. Sum of the radiation and Coulomb forces, $F^{(r)}_{x/y} + F^{(c)}_{x/y}$.

Convolution with Gaussian beam profile (E=8 GeV)

$\gamma\varepsilon_{x/y} = 1\ \mu$m, $\sigma_{x/y} = 25\ \mu$m, $\sigma_{p_{x/y}} = 2.5\ \mu$rad, $\sigma_z = 1\ \mu$m, $N_e = 10^{10}$.

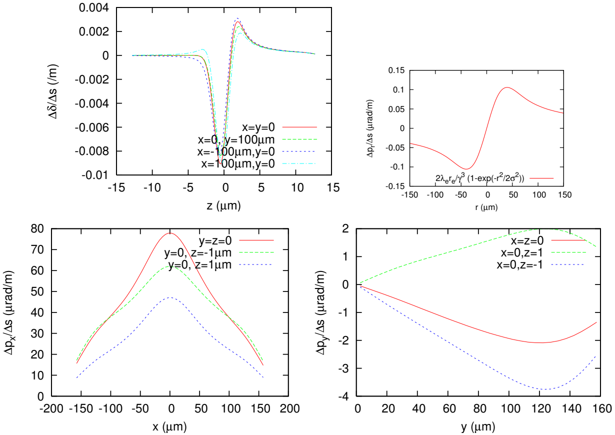

Figure: Convoluted beam kick. Sum. of radiation and Coulomb force. Top right is transverse space charge force in straight given by formula.

Convolution with Gaussian beam profile (E=1 GeV)

$\gamma\varepsilon_{x/y} = 0.125\ \mu$m, $\sigma_{x/y} = 25\ \mu$m, $\sigma_{p_{x/y}} = 2.5\ \mu$rad, $\sigma_z = 1\ \mu$m, $N_e = 10^{10}$.

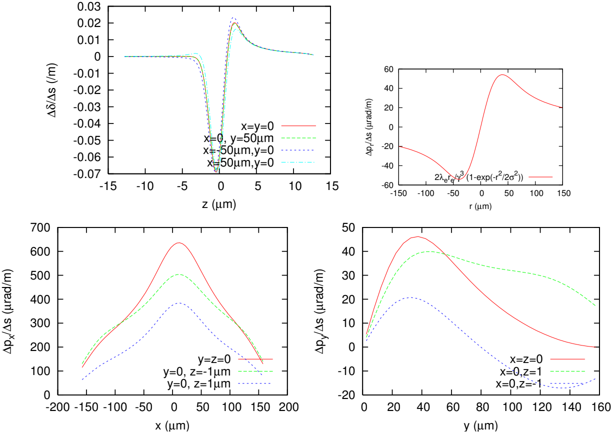

Figure: Convoluted beam kick. Sum. of radiation and Coulomb force. Top right is transverse space charge force in straight given by formula.

Convolution with Gaussian beam profile (E=1 GeV)

$\gamma\varepsilon_{x/y} = 1$ μm, $\sigma_{x/y} = 72$ μm, $\sigma_{p_{x/y}} = 7.2$ μrad, $\sigma_z = 1$ μm, $N_e = 10^{10}$.

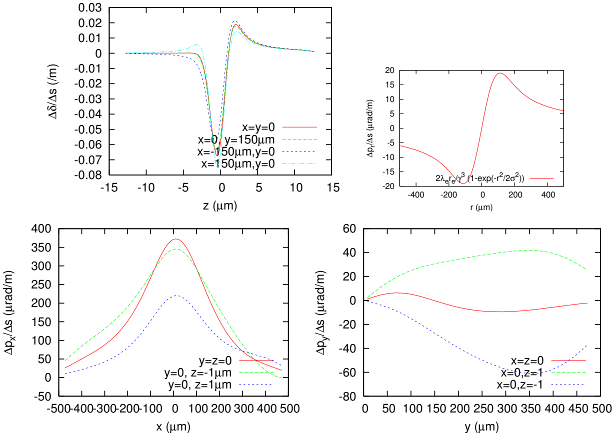

Figure: Convoluted beam kick. Sum. of radiation and Coulomb force. Top right is transverse space charge force in straight given by formula.

Convolution with Gaussian beam profile (E=0.2 GeV)

$\gamma\varepsilon_{x/y} = 0.025\ \mu$m, $\sigma_{x/y} = 25\ \mu$m, $\sigma_{p_{x/y}} = 2.5\ \mu$rad, $\sigma_z = 1\ \mu$m, $N_e = 10^{10}$.

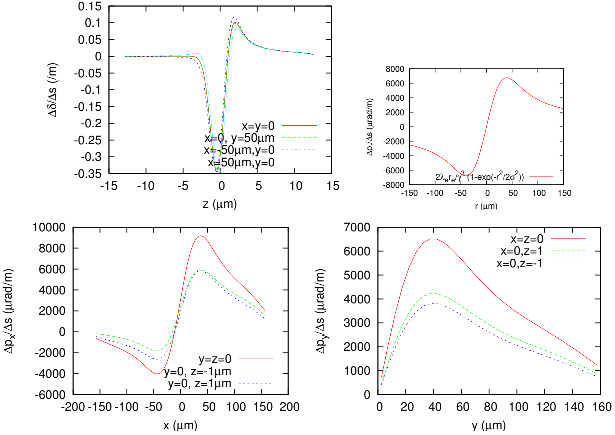

Figure: Convoluted beam kick. Sum. of radiation and Coulomb force. Top right is transverse space charge force in straight given by formula.

Convolution with Gaussian beam profile (E=0.2 GeV)

$\gamma\varepsilon_{x/y} = 1\ \mu\text{m},\ \sigma_{x/y} = 160\ \mu\text{m},\ \sigma_{p_{x/y}} = 16\ \mu\text{rad},\ \sigma_z = 1\ \mu\text{m},\ N_e = 10^{10}.$

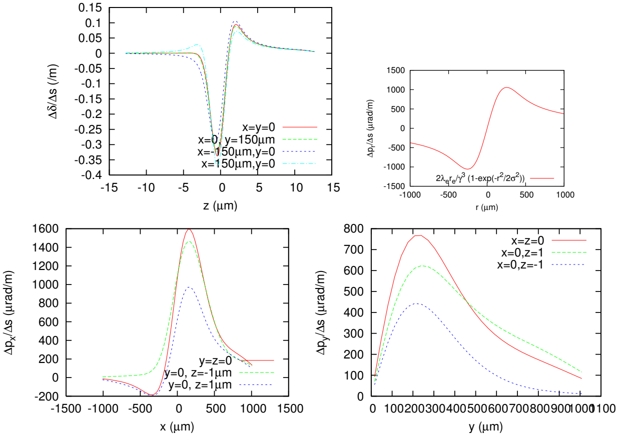

Figure: Convoluted beam kick. Sum. of radiation and Coulomb force. Top right is transverse space charge force in straight given by formula.

SACLA final branch line (E=8 GeV)

$\gamma\varepsilon_{x/y} = 0.8\ \mu$m, $\sigma_{x/y} = 23\ \mu$m, $\sigma_{p_{x/y}} = 2.3\ \mu$rad, $\sigma_z = 3\ \mu$m,
$N_e = 1.685 \times 10^9$ (270 pC).

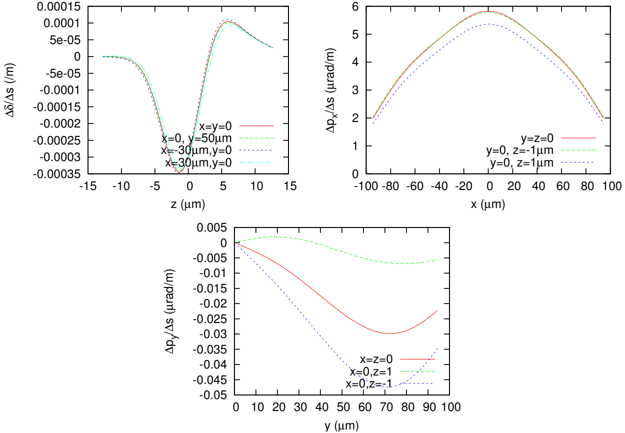

Figure: Convoluted beam kick. Sum. of radiation and Coulomb force. SACLA final branch line.

LCLS bunch compressor (E=0.22 GeV)

$\gamma\varepsilon_{x/y} = 0.5\ \mu$m, $\sigma_{x/y} = 170\ \mu$m, $\sigma_{p_{x/y}} = 6.9\ \mu$rad, $\sigma_z = 98\ \mu$m, $N_e = 1.29 \times 10^9$ (180 pC).

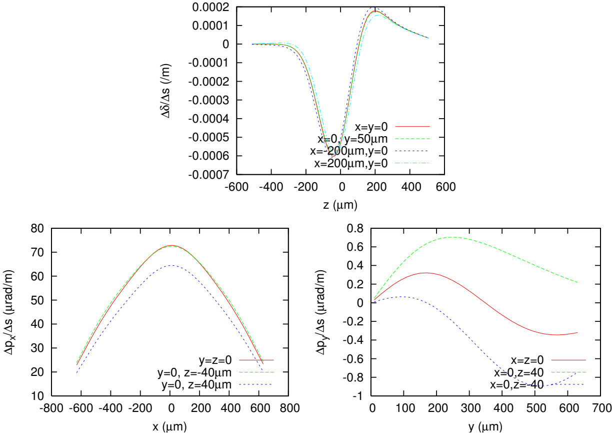

Figure: Convoluted beam kick. Sum. of radiation and Coulomb force. LCLS bunch compressor (E=0.22 GeV).

Entrance and Exit of bending magnet

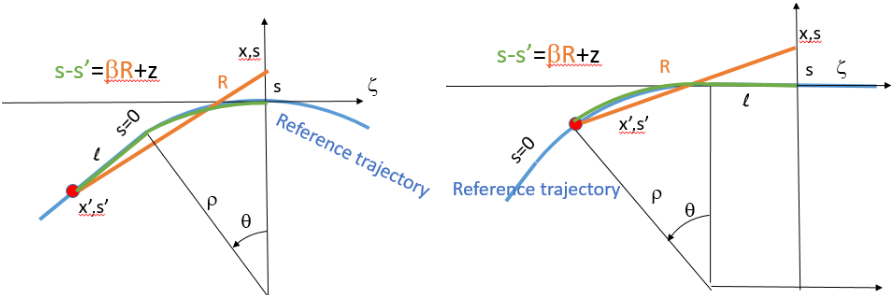

Figure: Definition of coordinates. x, y at Entrance and Exit

Entrance of bending magnet

$$\boldsymbol{x}' = (\rho(\cos\theta - 1) + \ell\sin\theta, 0, -\rho\sin\theta - \ell\cos\theta)$$

$$\boldsymbol{\beta}' = \beta(\sin\theta, 0, \cos\theta) \qquad \boldsymbol{\alpha}' = 0$$

$$z_\rho = \theta + \ell_\rho - \beta R_\rho$$

$$R_\rho = \sqrt{4(1+x_\rho)\sin^2\frac{\theta}{2} + x_\rho^2 + y_\rho^2 + \ell_\rho^2 + 2\ell_\rho(1+x_\rho)\sin\theta} \qquad (32)$$

$$\boldsymbol{n} = \frac{1}{R_\rho}\left(x_\rho + 2\sin^2\frac{\theta}{2} + \ell_\rho\sin\theta, y_\rho, \sin\theta + \ell_\rho\cos\theta\right)$$

$$\kappa = 1 - \boldsymbol{n}\cdot\boldsymbol{\beta}' = 1 - \frac{\beta\left\{(1+x_\rho)\sin\theta + \ell_\rho\right\}}{R_\rho} \qquad (33)$$

No serious numerical cancellation for $\ell_p > 1/\gamma$.

Exit of bending magnet

$$\boldsymbol{x}' = (\rho(\cos\theta - 1), 0, -\rho\sin\theta - \ell)$$

$$\boldsymbol{\beta}' = \beta(\sin\theta, 0, \cos\theta) \qquad \boldsymbol{\alpha}' = \frac{\beta^2}{\rho}(-\cos\theta, 0, \sin\theta)$$

$$z_\rho = \theta + \ell_\rho - \beta R_\rho$$

$$R_\rho \equiv \frac{R}{\rho} = \sqrt{4\,(1 + x_\rho)\sin^2\frac{\theta}{2} + x_\rho^2 + y_\rho^2 + \ell_\rho^2 + 2\ell_\rho\sin\theta} \tag{34}$$

$$\boldsymbol{n} = \frac{1}{R_\rho}\left(x_\rho + 2\sin^2\frac{\theta}{2}, y_\rho, \sin\theta + \ell_\rho\right)$$

$$\kappa = 1 - \boldsymbol{n}\cdot\boldsymbol{\beta}' = 1 - \frac{\beta\left\{(1 + x_\rho)\sin\theta + \ell_\rho\cos\theta\right\}}{R_\rho} \tag{35}$$

LW-field near a charged particle in undulator

$z = s - ct = \beta c t_0 - ct$: time advance for light arriving at s.

$$H = (1+\delta) - \sqrt{(1+\delta)^2 - \left(\boldsymbol{p} - \frac{\boldsymbol{a}}{\gamma}\right)^2 - \frac{1}{\gamma^2}} \tag{36}$$

$$\boldsymbol{a} = \frac{e}{mc}(A_x, A_y) \qquad a_x = -K \cos k_u s \quad a_y = 0$$

Equation of motion

$$\frac{dx}{ds} = \frac{p_x - a_x/\gamma}{p_s} \qquad \frac{dp_x}{ds} = 0 \tag{37}$$

$$\frac{dz}{ds} = 1 - \frac{1+\delta}{p_s} \qquad \frac{d\delta}{ds} = 0 \tag{38}$$

$$p_s \equiv \sqrt{(1+\delta)^2 - \left(\boldsymbol{p} - \frac{\boldsymbol{a}}{\gamma}\right)^2 - \frac{1}{\gamma^2}}$$

Electron motion in undulator (approximated)

$$
\begin{aligned}
x_u(s) &= \frac{K}{\beta\gamma k_u}\sin k_u s \qquad z_u(s) = -\frac{1+K^2/2}{2\gamma^2}s - \frac{K^2}{8\gamma^2 k_u}\sin 2k_u s \\
\beta_{x,u}(s) &= \frac{K}{\gamma}\cos k_u s \qquad \beta_{s,u}(s) = 1 - \frac{1+K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2}\cos 2k_u s \\
\alpha_{x,u}(s) &= -\frac{\beta K k_u}{\gamma}\sin k_u s \qquad \alpha_{s,u}(s) = \frac{\beta K^2 k_u}{2\gamma^2}\sin 2k_u s
\end{aligned}
\tag{39}
$$

$$\beta_x^2+\beta_s^2=\beta^2+\frac{1}{4\gamma^4}(1+K^2\cos^2 k_u s)^2$$

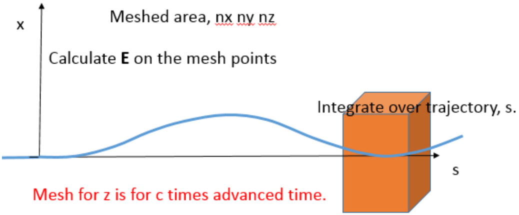

Figure: Integral in Undulator.

Retarded time condition

$$s - z = s' - z'(s') + R$$

Parameters of beam and undulator

E=8 GeV, $\gamma = 15289$, $L_u = 9$ m,$\lambda_u = 1.8$ cm, $k_u = 349$ m^{-1}, $K = 2$.

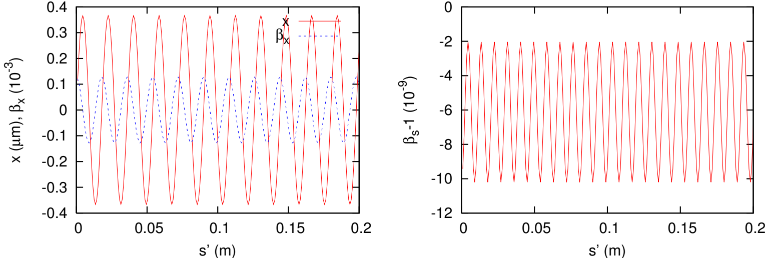

Figure: Particle motion for s'.

$|x| = K/\beta k_u \gamma = 0.366$ mm, $|\beta_x| = K/\gamma = 0.128 \times 10^{-3}$
$\langle \beta_s - 1 \rangle = (1 + K^2/2)/2\gamma^2 = 6.12 \times 10^{-9}$
$\lambda_r = 0.11$ nm.

Particle motion for z at $s = 0.9$ m

$z = 0$ is arrival time of light emitted at $s = 0$. Variables at $z < 0$
Arrival time of source particle with $z = 0$ at $s = 0$.

$$z \approx -\frac{1 + K^2/2}{2\gamma^2} s = -\frac{1}{2\gamma_z^2} s = -5.5 \text{ nm at } s = 0.9 \text{ m}$$

.

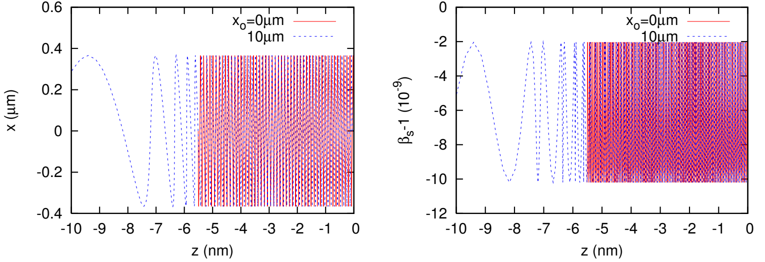

Figure: Particle motion, x' and β_s' for z at $x = 0, 10$ μm.

Electric field of undulator (preliminary)

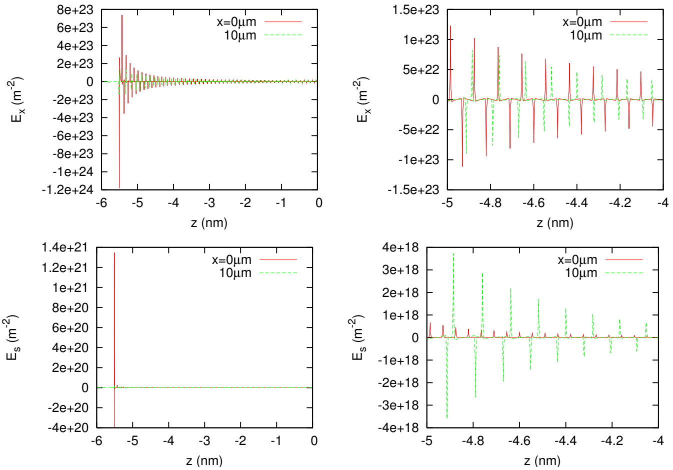

Figure: Electric fiel. E_x (top) and E_s (bottom)

$-z = 0$ is arrival time of light (reference).

$-z = 5.5$ nm is arrival time of source particle.

Lorentz force of undulator (preliminary)

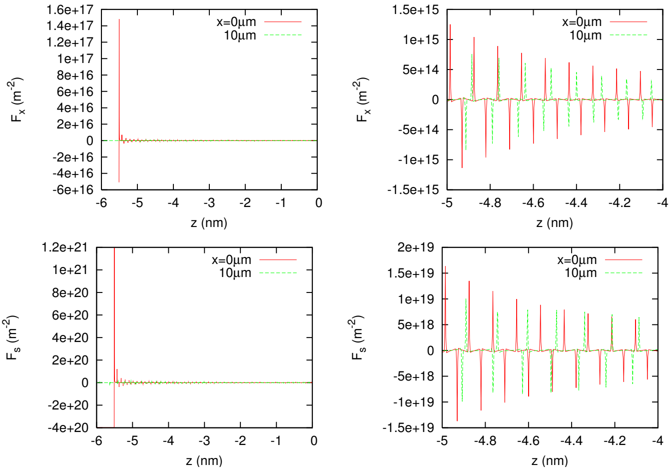

Figure: Lorentz force Electric fiel. F_x (top) and F_s (bottom).

Medium Q resonator, $Q \approx 10$. Decoherence for x. Large $s - s'$ is better coherence. Translation symmetry, $\boldsymbol{F}(z, s + d) \approx \boldsymbol{F}(z), d < \lambda_u$.

Wake field in undulator (preliminary)

Redefinition of z, $z - s/2\gamma_z^2 \Rightarrow z$,

$$\frac{d\boldsymbol{p}(\vec{x},s)}{ds} = \frac{Nr_e}{\gamma}\int \boldsymbol{F}(\vec{x},s;\vec{x}')\psi(\vec{x}'-\vec{x}_u(s),s)d\vec{x}' \qquad \text{for } \beta_{xy} > s-s' \quad (40)$$

$\psi(\vec{x}'-\vec{x}_u(s'),s') \approx \psi(\vec{x}'-\vec{x}_u(s),s)$ for $s-s' \ll L_{Gain}$.

Energy loss/gain during Δs, ($\lambda_u \ll \Delta s \ll L_{Gain}$).

$$\Delta\delta(\vec{x},s) = \int_{s-\Delta s}^{s} \boldsymbol{\beta}(s'')\cdot\frac{d\boldsymbol{p}(\vec{x},s'')}{ds''}ds'' \quad (41)$$

$$= \frac{Nr_e}{\gamma}\int \left[\int_{s-\Delta s}^{s} ds''\boldsymbol{\beta}(s'')\cdot\boldsymbol{F}(\vec{x},s'';\vec{x}')\right]\psi(\vec{x}'-\vec{x}_u(s),s)d\vec{x}'$$

$$\equiv \frac{Nr_e}{\gamma}\int F_\delta(\vec{x},\vec{x}',s)\psi(\vec{x}'-\vec{x}_u(s),s)d\vec{x}' \quad (42)$$

Translational symmetry, $\boldsymbol{F}(z,s+d) \approx \boldsymbol{F}(z+d/2\gamma_z^2,s)$ for $x_u \ll \sigma_x$ and new z. The integration for s'' is trivial.

Summary

1. Lienard-Wiechert potential/field near single particle was calculated along given trajectory in bending magnet.
2. Integrated Green (Wake) Functions for longitudinal/transverse fields were obtained.
3. The transverse force may have some effects depending on parameters.
4. Longitudinal and transverse field in undulator were obtained.

Next steps

1. Entrance and Exit of bending magnet
2. Regularization/Integrated Green Function in undulator.
3. Simulate bunch motion using the wake field.

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The End

Thank you for your attention

Electron motion in undulator

Solution

$$p_x = \delta = 0 \quad \rightarrow \quad p_s = \sqrt{1 - \frac{1 + K^2 \cos^2 k_u s}{\gamma^2}} \approx \sqrt{1 - \frac{1 + K^2/2}{\gamma^2}} \equiv \beta$$

$$\frac{dx}{ds} \approx \frac{K}{\beta\gamma} \cos k_u s \qquad x = \frac{K}{\beta\gamma k_u} \sin k_u s \tag{43}$$

$$\frac{dz}{ds} = 1 - \frac{1}{\sqrt{1 - \frac{1+K^2 \cos^2 k_u s}{\gamma^2}}} \approx -\frac{1 + K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2} \cos 2k_u s$$

$$z \approx -\frac{1 + K^2/2}{2\gamma^2} s - \frac{K^2}{8\gamma^2 k_u} \sin 2k_u s \tag{44}$$

x, β, α

$$\beta_x = \frac{dx}{dct} = \frac{dx}{ds}\left(\frac{dct}{ds}\right)^{-1} \approx \frac{K}{\gamma}\cos k_u s \tag{45}$$

$$\alpha_x = \frac{d\beta_x}{dct} = \frac{d\beta_x}{ds}\left(\frac{dct}{ds}\right)^{-1} = -\frac{\beta K k_u}{\gamma}\sin k_u s \tag{46}$$

$$\left(\frac{dct}{ds}\right)^{-1} = \left(1 - \frac{dz}{ds}\right)^{-1} = p_s \approx 1 - \frac{1 + K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2}\cos 2k_u s$$

$$\beta_s = \left(\frac{dct}{ds}\right)^{-1} = 1 - \frac{1 + K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2}\cos 2k_u s \tag{47}$$

$$\alpha_s = \frac{d\beta_s}{dct} = \frac{d\beta_s}{ds}\left(\frac{dct}{ds}\right)^{-1} = \frac{\beta K^2 k_u}{2\gamma^2}\sin 2k_u s \tag{48}$$

Particle motion for z at $s = 0.9$ m

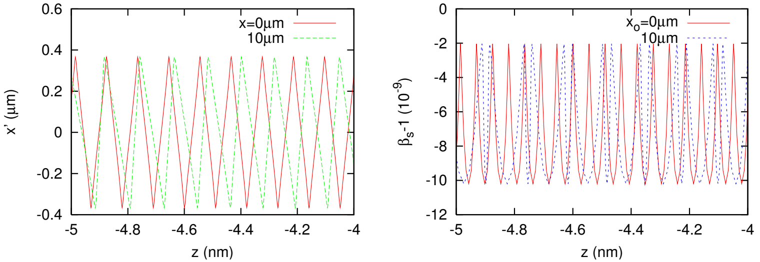

Figure: Particle motion, x' and β_s' for z.