Mode expansion method for calculation of CSR impedance

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Outline

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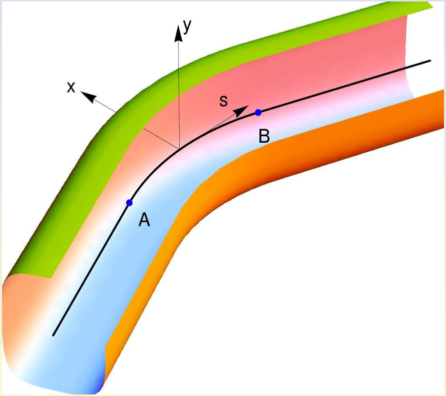
- Parabolic equation and general formulation of the problem
- Comparison of my code with numerical results of DZ
- Different approach to the mode expansion method and some new results
- CUR impedance in rectangular pipe
- Conclusions

Using parabolic equation for CSR calculations

- In 2009 [*Stupakov and Kotelnikov, PRST-AB*, 2009] we used PE and mode expansion method to compute the wake of a toroidal segment with infinitely long incoming and outgoing pipes of rectangular cross section.
- In a typical case, the characteristic transverse size of the vacuum chamber a is much smaller than the bending radius R, $a \ll R$.
- The small parameter $\epsilon = \sqrt{a/R}$ can be used to simplify Maxwell's equations, keeping only terms to the lowest order in ϵ. In this approximation the transverse components of the electric field satisfy a so called *parabolic equation*.
- We assumed perfect conductivity of the walls and relativistic particles with the Lorentz factor $\gamma = \infty$.

CSR in a toroidal cegment (bending magnet of finite length)

Vacuum chamber has a smooth toroidal segment of radius R and of arbitrary cross section, connected to two straight pipes.



The characteristic transverse dimension of the pipe is a. The coordinate along the axis of the toroid is s with $s = 0$ at the entrance A. The cylindrical coordinates are r and y and $x = r - R$.

The beam initially carries Coulomb field in the straight pipe, enters the toroidal segment, travels in it, and then exits into the straight pipe again.

Fourier transformation

The Fourier transformed components of the field and the current defined as

$$\hat{\mathbf{E}}(x, y, s, \omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t - iks}\, \mathbf{E}(x, y, s, t)\,,$$
$$\hat{j}_s(x, y, s, \omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t - iks}\, j_s(x, y, s, t)\,,$$

where $k \equiv \omega/c$, and j_s is the projection of the beam current onto s. The transverse component of the electric field $\hat{\mathbf{E}}_\perp$ is a two-dimensional vector $\hat{\mathbf{E}}_\perp = (\hat{E}_x, \hat{E}_y)$. The longitudinal component is denoted by $\hat{E}_s$.

Parabolic equation for the field

A mathematical assumption that leads to the parabolic equation is a slow dependence of the functions $\hat{\mathbf{E}}_\perp$ and $\hat{j}_s$ versus s, such that $\partial/\partial s \ll k$:

$$\frac{\partial}{\partial s}\hat{\mathbf{E}}_\perp = \frac{i}{2k}\left(\nabla_\perp^2 \hat{\mathbf{E}}_\perp + \frac{2k^2 x}{R}\hat{\mathbf{E}}_\perp - \frac{4\pi}{c}\nabla_\perp \hat{j}_s\right)$$

$\nabla_\perp = (\partial/\partial x, \partial/\partial y)$.
The longitudinal electric field can be expressed through the transverse one and the current,

$$\hat{E}_s = \frac{i}{k}\left(\nabla_\perp \cdot \hat{\mathbf{E}}_\perp - \frac{4\pi}{c}\hat{j}_s\right)$$

Boundary conditions:

$$\hat{\mathbf{E}}_\perp\Big|_w \times \mathbf{n} = 0, \qquad \hat{E}_s\Big|_w = 0$$

The second equation reduces to $(\operatorname{div}\hat{\mathbf{E}}_\perp)\Big|_w = 0$ on the wall.

Eigenmodes of toroidal rectangular waveguide

Eigenmodes are solutions of the parabolic equations with $\hat{j}_s = 0$.
Each eigenmode can be characterized by two integer indices, m and p, and the wavenumber $q_{mp}(\omega)$, which is a function of the frequency ω,

$$\hat{\mathbf{E}}_{mp,\perp}(x, y, s) = \mathcal{E}_{mp,\perp}(x, y)\, e^{iq_{mp}(\omega)s},$$
$$\hat{E}_{mp,s}(x, y, s) = \mathcal{E}_{mp,s}(x, y)\, e^{iq_{mp}(\omega)s}\ .$$

The parabolic equation is applicable if $|q_{mp}| \ll \omega/c$. These modes constitute a set of orthogonal functions.

If $q_{mp} = 0$, the mode has phase velocity equal to c and is resonant with the beam ($\gamma = \infty$).

In general case, for a given transverse shape of the pipe, finding eigenmodes represents a two dimensional problem which can be solved numerically. For a pipe with rectangular cross section eigenmodes can be found analytically.

Field expansion

Consider a point charge moving with a speed of light along the axis. We expand the perpendicular part of the electric field $\hat{\mathbf{E}}_\perp$ generated by the current j_s into the series

$$\hat{\mathbf{E}}_\perp = \sum_{p,m} C_{mp}(s)\hat{\mathbf{E}}_{mp,\perp}(x,y,s)$$

over the eigenmodes. Equation for the series coefficients:

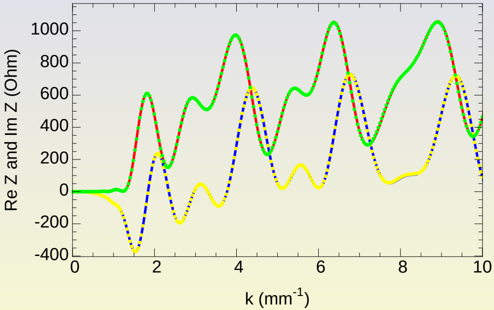
$$\frac{dC_{mp}}{ds} = -\frac{2\pi i}{\omega}\, e^{-iq_{mp}s} \iint dx\, dy \left(\nabla_\perp \hat{j}_s \cdot \mathcal{E}^*_{mp,\perp}\right)$$

The above equations describe the field in the toroidal segment. At the exit point B from the segment, we re-expand the field into the eigenmodes of the straight rectangular pipe (also computed within the paraxial approximation) and find the beam field in the exit pipe.

We wrote a Mathematica code that computes the longitudinal field $\hat{E}_s(s,\omega)$ on the orbit for a given *rectangular* geometry.

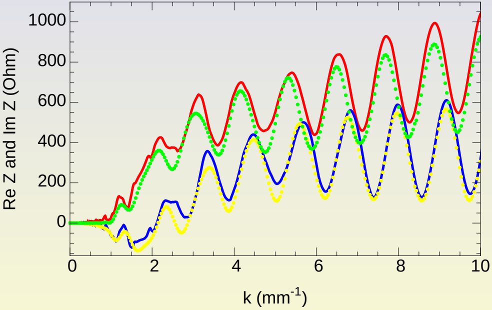
Comparison with DZ

Bending magnet with $\rho = 16.3$ m, $L = 4$ m, rectangular pipe, full height 40 mm, pipe width 60 mm. Beam is in the center of the pipe.



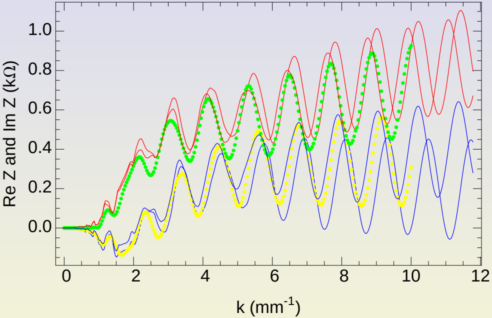
Beam off-center

The same magnet, but the beam is at 10 mm from the inner wall—not so good agreement. This is most likely due to the finite transverse size ($\sigma_\perp = 0.5$ mm) in DZ calculations.



Beam off-center

Offsets 10±0.5 mm, comparison with DZ.



New approach to the mode expansion method

In this approach I use eigenmodes $\mathcal{E}_{mp,\perp}(x,y)\, e^{i\tilde{q}_{mp}s}$ of the *straight* pipe (of the given cross section) and a solution $\hat{\mathbf{E}}_{\perp}^{ss}(x,y)$ for the field of the point charge moving with $v=c$ in the *straight* pipe along the axis [a rapidly converging infinite series].

In the toroidal section we expand the perpendicular part of the electric field $\hat{\mathbf{E}}_{\perp}$

$$\hat{\mathbf{E}}_{\perp} = \hat{\mathbf{E}}_{\perp}^{ss}(x,y) + \sum_{p,m} C_{mp}(s)\tilde{\mathbf{E}}_{mp,\perp}(x,y,s)$$

One can obtain

$$\frac{dC_{mp}}{ds} = \frac{ik}{R(s)}\, e^{-i\tilde{q}_{mp}s} \iint dx\, dy \left(x\hat{\mathbf{E}}_{\perp}^{ss} \cdot \mathcal{E}_{mp,\perp}^{*}\right) + \frac{ik}{R(s)} \sum_{p',m'} C_{m'p'}(s)\; e^{i(\tilde{q}_{m'p'}-\tilde{q}_{mp})s} \iint dx\, dy \left(x\mathcal{E}_{m'p',\perp} \cdot \mathcal{E}_{mp,\perp}^{*}\right)$$

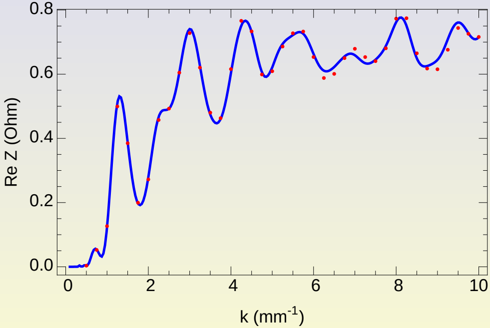
Advantages and disadvantages

- The singularity due to the field of the point charge is eliminated (absorbed in $\hat{\mathbf{E}}_{\perp}^{ss}$ and integrated out).
- Standard ODE solvers can be used for solution of the differential equations.
- Arbitrary $R(s)$ can be treated—many magnets with straights between them.
- Other cross-sections for which analytical expression for $\hat{\mathbf{E}}_{\perp}^{ss}$ and $\mathcal{E}_{\mathrm{mp},\perp}(x, y)$ exist can be treated (round, elliptical, ...).
- The code is slower.

To expedite code development, I only coded the part which calculates $\mathrm{Re}\, Z$.

Two magnets with rectangular cross section

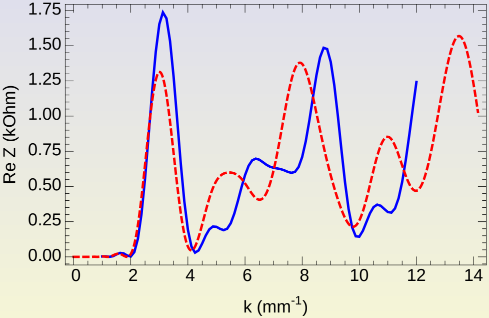
$R_1 = 2.68$ m, $L_1 = .742$ m, $L_{drift} = .927$ m, $R_2 = -2.958$ m, $L_2 = .286$ m. Cross section of the beam pipe: square with width/height = 34/34mm.



Solid curve - DZ, dots - GS.

Round cross section of the pipe

$R = 16.3$ m, $L = 4$ m, round pipe $r = 20$ mm. Comparison with a square one with the same cross section area $a = 35.4$ mm.



Wiggler CUR impedance in rectangular waveguide

CUR impedance inside a rectangular waveguide was studied by Y.-H. Chin in preprint LBL-29981, 1990. Unfortunately, he only considered the limit of a weak undulator, $K \ll 1$.
In the opposite limit, $K \gg 1$, several simplifying approximations can be made:

- If we are interesed in the wavelengths much longer then the fundamental wavelength, $\lambda \gg \lambda_0$, we can assume $v = c$.
- Moreover, one can approximate $v_z = c$.
- The amplitude of trajectory wiggling $\ll$ transverse size of the pipe.

Calculating $\mathrm{Re}\, Z$ through radiated power

One can calculate the spectral power of radiation of a point charge $P(\omega)$ moving in the undulator and relate it to the real part of the longitudinal impedance

$$\mathrm{Re}\, Z(\omega) = \frac{\pi}{q^2} P_\omega$$

This is not a complete solution of the wakefield problem, but it is good enough for comparison and benchmarking codes. Note that the Kramers-Kronig relations between $\mathrm{Re}\, Z$ and $\mathrm{Im}\, Z$ do not hold in general (but they may hold in my approximation[?]).
I do not use the paraxial approximation in this problem, which allows for checking the accuracy of the parabolic equation (used in DZ code).

Calculating radiating power

Working in Fourier representation, everything $\propto e^{-i\omega t}$. Assume radiation in the forward direction only, $\mathbf{E}_n^+$

$$\mathbf{E}^{\mathrm{rad}} = \sum_n a_n \mathbf{E}_n^+, \qquad a_n = -\frac{1}{N_n}\int \mathbf{j}\cdot\mathbf{E}_n^- dV$$

$$N_n = \frac{c}{4\pi}\int \left(\mathbf{E}_n^+ \times \mathbf{H}_n^- - \mathbf{E}_n^- \times \mathbf{H}_n^+\right)\cdot d\mathbf{S}$$

$$P_\omega = \frac{2}{\pi}\sum_n P_n |a_n|^2,$$

where P_n is the energy flow in the mode of unit amplitude ($P_n = N_n/4$).

Wiggler CUR impedance

CUR in free space in the limit $K \gg 1$ was previously studied by Wu, Stupakov and Raubenheimer [PRST-AB, 6, 040701 (2003)]. In the limit of low frequencies, $k \ll k_0$, k_0 is the fundamental radiation wavenumber

$$Z(k) = \pi k \frac{k_w}{k_0} \left(1 - \frac{2i}{\pi} \log \frac{k}{k_0} \right)$$

(the impedance per unit length).

Analytical result for $\operatorname{Re} Z$

Assume a sinusoidal orbit, $x(z) = (\theta_0/k_w)(1 - \cos k_w z)$ in an undulator with N_u periods. The result is

$$\operatorname{Re} Z(k) = 4 Z_0 \theta_0^2 F(k),$$

$$F(k) = \frac{k_w^2}{abk} \sum_{n_1, n_2} \left(\frac{k^2 {k_y}^2 (2 - \delta_{0,n_1})}{2 k_z \varkappa^2} + \frac{k_x^2 \varkappa^2}{k_z} \left(\frac{k_z}{\varkappa^2} - \frac{1}{k - k_z} \right)^2 \right)$$
$$\times \frac{\sin^2[\pi N_u (k - k_z)/k_w]}{[(k - k_z)^2 - k_w^2]^2}, \quad k_x = \frac{n_1}{a}, \, k_y = \frac{n_2}{b},$$
$$\varkappa^2 = k_x^2 + k_y^2, \, k_z = \sqrt{k^2 - \varkappa^2}$$

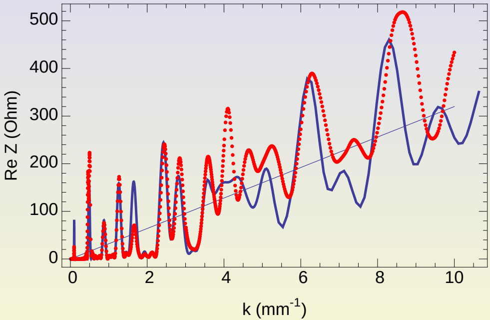
In the limit $N_u \to \infty$

$$\frac{\sin^2[\pi N_u (k - k_z)/k_w]}{[(k - k_z)^2 - k_w^2]^2} \to \frac{\pi^2}{k_w^3} N_u \delta\left(k - k_z - k_w\right)$$

Expect narrow peaks at $k - k_z - k_w = 0$.

KEK-B wiggler impedance, comparison with DZ

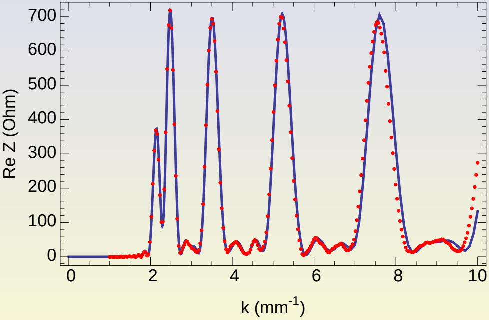
Sinusoidal wiggler: wiggler period: 1.08762m, $K = 76.57$, $\gamma = 6850$, $N_u = 10$, pipe width/height: 94/94mm



DZ uses his CSR code with sinusoidal $R(s)$: the code assumes wiggling vertical wall, that follows the shape of the beam trajectory.

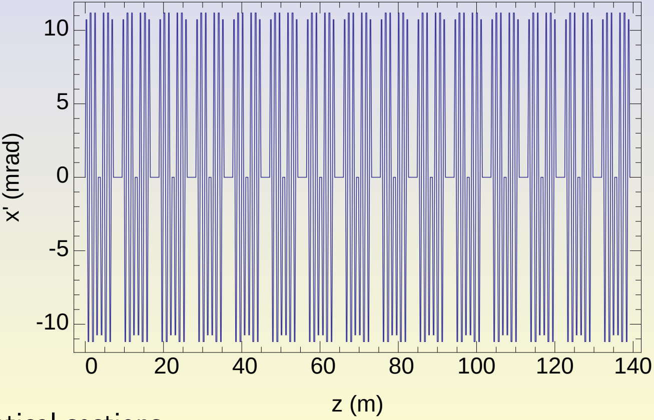
KEK-B wiggler impedance, comparison with DZ

The same wiggler as above, but the pipe is 100 mm$\times$ 20 mm.
Perfect agreement with DZ.



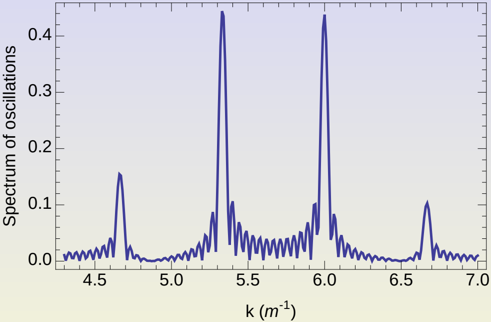
SuperKEKB wiggler orbit

We used the magnetic field of the wiggler to compute particle's orbit



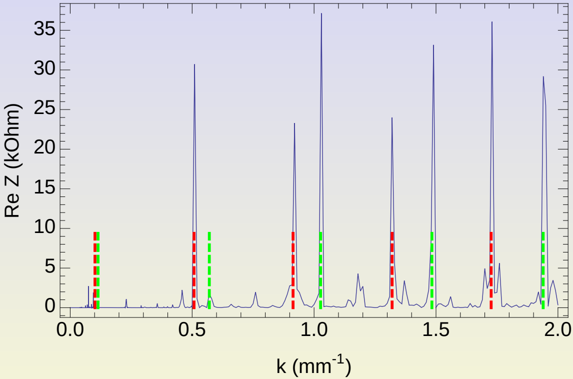
15 identical sections

Spectrum of the orbit



There are two distinct periods with $k \approx 6\ \mathrm{m}^{-1}$ and $k \approx 5.34\ \mathrm{m}^{-1}$.

SuperKEKB wiggler CUR impedance



Pipe: 90×90 mm. Peaks are the modes with $k_z(n_1, n_2) = k - k_w$: red - $k \approx 6$ m^{-1}, green - $k \approx 5.34$ m^{-1} $(n_1, n_2 < 4)$.

Conclusions

- Comparison of the mode expansion code (GS) with numerical ones (DZ) shows very good agreement in a simple case of one magnet.
- The modified mode expansion approach allows to treat multiple magnets, as well as pipe cross-sections different from rectangular (round).
- Analytical results for the $\mathrm{Re}Z$ of the CUR are derived.
- SuperKEK-B wiggler impedance demonstrates sharp narrow peaks in the range of sub-centimeter wavelengths.