New approach to numerical solution of the microwave instability

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Motivation

- Analysis of the beam instabilities is an important part of an accelerator design. In the design phase one is mostly interested in the threshold of the instabilities. Additional information can be also obtained from the knowledge of the growth rate of the instability. This means that a full simulation of the beam dynamics is not needed—it is enough to solve a linearized system of equations in the vicinity of the equilibrium state of the beam.

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- Solution of linearized Vlasov equation for MWI was first implemented in a code by Oide&Yokoya in 1990, and several important results were obtained in the past with that code. Unfortunately, the code does not always give reliable results. In SLAC we rely on (nonlinear) Vlasov-Fokker-Planck solver originally developed by R. Warnock.

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- In this talk I will advocate for a different approach to solution of the linearized Vlasov (LV) equation which uses a simple discretization scheme and some well developed, standard computational tools.

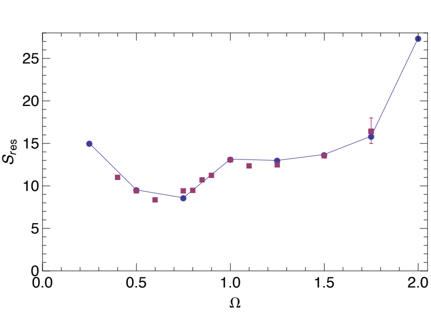
Outline of the talk

- Introduction and a recent paper on MWI.
- Coasting beam model and the Keil-Schnell stability criterion. This is exactly solvable and can be used for benchmarking new algorithms.
- Coasting beam with periodic boundary conditions, discretization of the LVE and comparison with the continuous coasting beam.
- LVE for a bunched beam and its discretization.
- An example of numerical solution for the bunched beam and challenges.

Recent study of MWI

In a recent paper (K. Bane, Y. Cai and G. Stupakov, PRST-AB, Oct. 2010) we studied MWI with the goal to compare a full (nonlinear) Vlasov-Fokker-Planck solver (VLF code by Y. Cai, based on the original code from R. Warnock) and a LV code (G. Stupakov).

Threshold of the MW instability for a broadband resonant impedance with $Q = 1$.



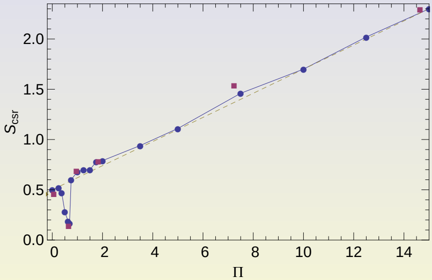
LV code (red), VFP code (blue).

$$S = W(0)\frac{r_e N_b}{2\pi\nu_{s0}\gamma\sigma_{\eta 0}}$$

$$\Omega = \frac{\sigma_{z0}\omega_r}{c}$$

CSR induced microwave instability

CSR impedance with shielding by parallel conducting plates.
Threshold of the MWI instability is a function of the shielding parameter Π.



LV code (red), VFP code (blue and olive).

$$S = \rho^{1/3}\sigma_{z0}^{-4/3}\frac{r_e N_b}{2\pi\nu_{s0}\gamma\sigma_{\eta 0}}$$

$$\Pi = \frac{\sigma_{z0}\rho^{1/2}}{h^{3/2}}$$

h is the half-gap between the plates, ρ is the bending radius

Infinitely long coasting beam

The beam occupies the region $-\infty < z < \infty$, has some energy spread, and is uniform in equilibrium (no z-dependence).
The distribution function $f(t, z, \eta)$ satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} - c\mu\eta\frac{\partial f}{\partial z} + K\frac{\partial f}{\partial \eta} = 0$$

μ the slip factor (assumed positive), η is the relative energy deviation, and z is the longitudinal coordinate in the beam. $\int d\eta f$ is equal to the number of particles per unit length.

$$K(t, z) = -\frac{cr_e}{\gamma}\int_{-\infty}^{\infty} dz' w(z' - z)\int_{-\infty}^{\infty} d\eta f(t, z', \eta)$$

with γ the relativistic factor and w the wake per unit length of path. Positive values of w correspond to the energy loss.

Perturbation and linearization

Assume an equilibrium distribution function with rms spread σ_η

$$f_0(\eta) = \frac{1}{\sigma_\eta} F_0\left(\frac{\eta}{\sigma_\eta}\right), \qquad F_0(x) = (2\pi)^{-1/2} e^{-x^2/2}$$

In equilibrium $K_0 = 0$. Assume a small deviation from the equilibrium, $f = f_0 + f_1$, $K = K_1$. Introduce dimensionless variable $p = -\eta/\sigma_\eta$.

Linearized Vlasov equation

$$\frac{\partial f_1}{\partial t} + c\mu\sigma_\eta p \frac{\partial f_1}{\partial z} - \frac{1}{\sigma_\eta} K_1 \frac{\partial f_0}{\partial p} = 0$$

with

$$K_1(t,z) = -\frac{c r_e \sigma_\eta}{\gamma} \int_{-\infty}^{\infty} dz' w(z'-z) \int_{-\infty}^{\infty} dp f_1(t,z',p)$$

Dispersion relation

Consider a perturbation with the wavenumber k and the frequency ω,

$$f_1(t, z, p) = \phi(p)e^{-i\omega t+ikz}, \qquad \hat{K}_1 = Qe^{-i\omega t+ikz}.$$

Introduce the longitudinal impedance $Z(k)$

$$Z(k) = \frac{1}{c}\int_{-\infty}^{\infty} d\xi w(\xi)e^{ik\xi}$$

The dispersion relation

$$1 = i\frac{cr_e}{\gamma\mu\sigma_\eta^2}\frac{Z(k)}{k}\int_{-\infty}^{\infty}\frac{\partial F_0/\partial p}{p-a},$$

where $a = \omega/c\mu k\sigma_\eta$. From this equation, for a given k, one finds a and $\omega = ac\mu k\sigma_\eta$ (in general complex).

Dispersion relation

The function

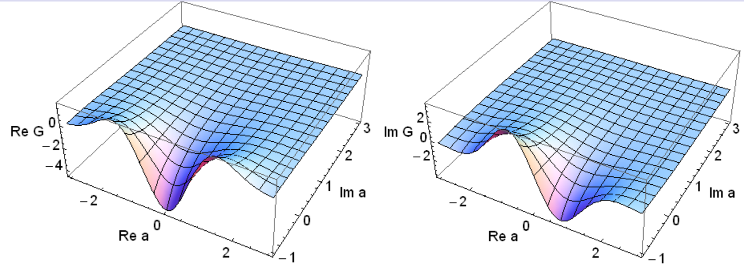
$$G(a) = \int_{-\infty}^{\infty} \frac{\partial F_0/\partial p}{p - a}$$

is defined in the complex plane of variable a.

A more accurate derivation that uses the Laplace transform instead of the Fourier one shows that the integral representation is valid in the upper half-plane of the complex variable a $(\mathrm{Im}\,\omega > 0)$, and should be analytically continued to the lower half-plane. It turns out that the result (for a Gaussian F_0) is expressed in terms of the error function of a complex argument.

Dispersion functions

Plot of the function $G(a)$.



Broadband resonant impedance

I will use the broadband resonator wake. It depends on two parameters, Q and ω_0, and for positive z is

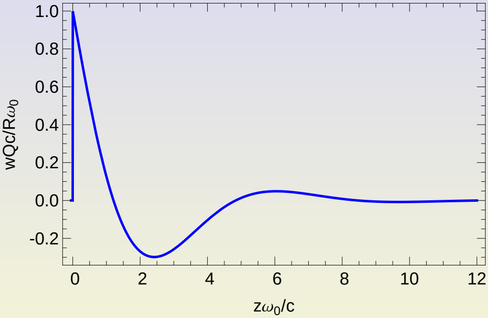
$$w(z) = \frac{R\omega_0}{Q} e^{-z\omega_0/2cQ} \left(\cos(z\omega_1/c) - \frac{\sin(z\omega_1/c)}{\sqrt{4Q^2 - 1}} \right)$$

with $w = 0$ for $z < 0$ and $\omega_1 = \omega_0/\sqrt{1 - 1/(4Q^2)}$. The quantity R is the shunt impedance per unit length.
The impedance

$$Z(k) = \frac{R}{1 + iQ(\omega_0/kc - kc/\omega_0)}$$

Broadband impedance

In all calculations I used $Q = 1$.



Dispersion equation for the broadband impedance

Using dimensionless wavenumber $\varkappa = kc/\omega_0$ the dispersion relation can be written as

$$1 = iS\frac{Z(\varkappa)/R}{\varkappa}G(a)$$

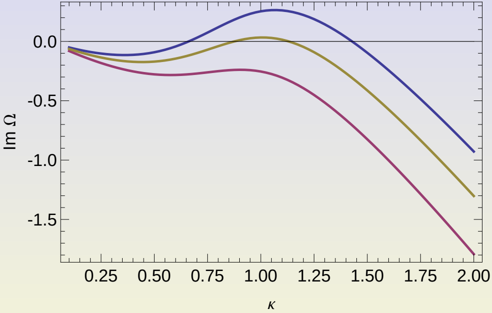
with

$$S = \frac{r_e I_b R c}{e\gamma\mu\sigma_\eta^2\omega_0}$$

where I_b is the beam current. The dimensionless frequency $\Omega = a\varkappa = \omega/\omega_0\mu\sigma_\eta$ is the frequency ω measured in units of $\omega_0\mu\sigma_\eta$.

Dispersion curves

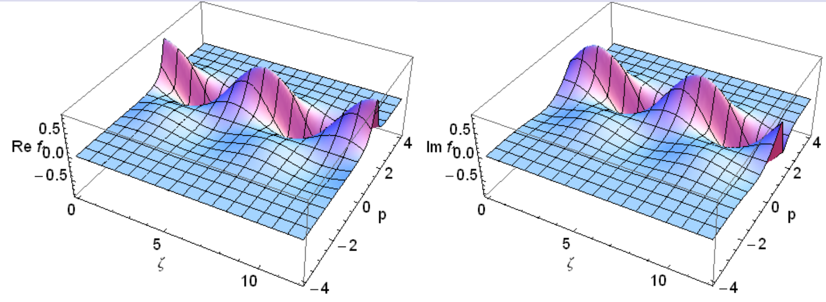
The solutions of the dispersion relation for $S = 1,\ 1.5,\ 2$.



The Landau damping suppresses the instability for a small currents and/or short wavelengths. The threshold of the instability is at $S \approx 1.45$.

Unstable eigenmodes

The eigenmodes for $S = 2$ and $\kappa = \pi/3$.

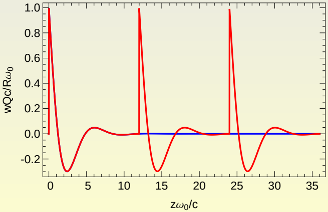


Periodic system

For a numerical solution we need to make the length of the system finite. Impose periodicity with a period L, that is $f(t, z + nL, \eta) = f(t, z, \eta)$ for any integer n. Also introduce periodic wake

$$\tilde{w}(z) = \sum_{k=-\infty}^{\infty} w(z + kL)$$

If the period is long, results should be close to that of the infinite system. I choose $L = 12c/\omega_0$.



Periodic system

In the LVE K_1 becomes a periodic function of z

$$K_1(t,z) = -\frac{c r_e \sigma_\eta}{\gamma} \int_0^L dz' \tilde{w}(z'-z) \int_{-\infty}^{\infty} dp f_1(t,z',p).$$

Our analysis for the infinite system can now be repeated for the periodic one with the only constrain that *the wave number* k *now takes discreet values only*, $k = 2\pi n/L$, where n is an integer. The stability condition does not change much.

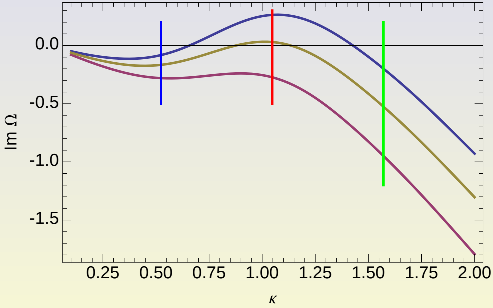
Correspondingly, the impedance now is

$$\tilde{Z}(k) = \frac{1}{c} \int_0^L d\xi \tilde{w}(\xi) e^{ik\xi}$$

If L is large, it is almost the same as for the infinite system.

Solution for the periodic system

The solution for the periodic system (for our parameters) is almost the same as for the continuous one, but for $\varkappa = \pi/6,\ \pi/3,\ 2\pi/3, \ldots$.



The solutions of the dispersion relation for $S = 1,\ 1.5,\ 2$.

Dimensionless variables

Introduce dimensionless variables $\zeta = z\omega_0/c$, $l = L\omega_0/c = 12$, and $\tau = t\mu\sigma_\eta\omega_0$ and transform the equations to the dimensionless variables

$$\frac{\partial f_1}{\partial \tau} + p\frac{\partial f_1}{\partial \zeta} + \hat{K}_1\frac{\partial F_0}{\partial p} = 0,$$

where $\hat{K}_1 = -(I/ec\sigma_\eta^3\mu\omega_0)K_1$

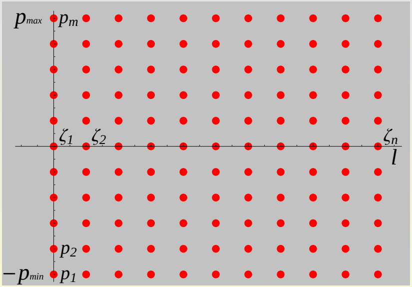
$$\hat{K}_1(\tau,\zeta) = S\int_0^l d\zeta' W(\zeta'-\zeta)\int_{-\infty}^{\infty} dp f_1(\tau,\zeta',p),$$

where we introduced the dimensionless wake $W = (Qc/R\omega_0)\tilde{w}$.

Discretization of the periodic linearized equation

Make a mesh in ζ with the mesh size $\Delta\zeta = l/n$ and the nodes $\zeta_1, \zeta_2, \ldots \zeta_n$ such that $\zeta_1 = 0$ and $\zeta_n = l - \Delta\zeta$ (due to periodicity, $\zeta_{n+1} = l$ is the same as $\zeta = 0$).

The mesh in p is $p_1, p_2, \ldots, p_m$, with the mesh size Δp, uniformly divides the interval from $-p_{\max}$ to $p_{\max}$.



The function f_1 taken at the nodes is denoted $f_{i,j}$, where the first index denotes position in ζ and the second one in p; analogously the derivative $\partial F_0/\partial p$ taken at p_j is denoted $F'_{0,j}$. The function $\hat{K}_1$ taken at ζ_i is denoted by $\hat{K}_{1i}$ and the discretized wake $W_i = W(\zeta_i)$.

Discretization of the periodic linearized equation

Discretized system

$$\frac{\partial}{\partial \tau} f_{i,j} = -p_j \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta\zeta} - \hat{K}_{1i} F'_{0,j}.$$

and

$$\hat{K}_{1i} = S \Delta p \Delta z \sum_{l=1}^{n} W_{l-i} \sum_{q=1}^{m} f_{l,q}$$

(one can also use Simpson method, or any other weighted approximation, for numerical for integrals).
This is a system of mn ODEs with constant coefficients. Its solution (if not degenerate) consists of mn eigenmodes with time dependence $e^{\lambda\tau}$. *The most unstable solution has the largest* $\mathrm{Re}\,\lambda$.

$$\lambda f_{i,j} = -p_j \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta\zeta} - \hat{K}_{1i} F'_{0,j}.$$

Analysis of the discretized system

Eigenfunctions of the discretized system depend on ζ via $e^{i\varkappa_r \zeta_i}$ (with $\varkappa_r = 2\pi r/l$, $r = 0, 1, 2, \ldots n-1$).

$$f_{i,j} = A_j e^{i\varkappa_r \zeta_i}, \qquad \hat{K}_{1,i} = B e^{i\varkappa_r \zeta_i},$$

For each $\varkappa_r$ this system has m eigenvalues Ω. This splits mn eigenvalues into n sets of m eigenvalues.

Substitute into equations and and replace $\lambda \to -i\Omega$. The discretized dispersion relation

$$1 = S\Delta p\Delta z \sum_l W_l e^{i\varkappa_r z_l} \sum_j \frac{F'_{0,j}}{i(\Omega - p_j \frac{\sin \varkappa_r \Delta\zeta}{\Delta\zeta})}$$

Discretized dispersion relation

It can be written similar to the continuous case

$$1 = iS\frac{Z_D(\varkappa_r)/R}{\varkappa_r} G_D\left(\frac{\Omega}{\varkappa_r}\right)$$

with

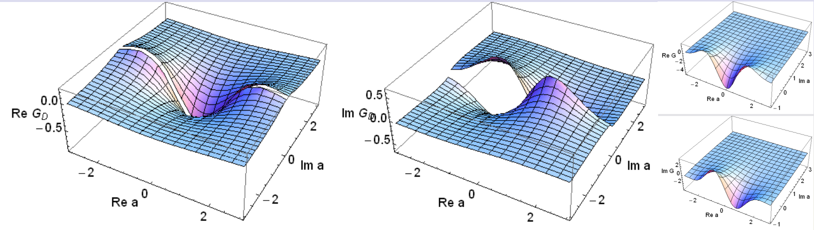
$$Z_D(\varkappa_r)/R = \Delta p \sum_l W_l e^{i\varkappa_r z_l}$$

$$G_D(a) = \Delta z \sum_j \frac{F'_{0,j}}{p_j \frac{\sin \varkappa_r \Delta\zeta}{\Delta\zeta\varkappa_r} - a}$$

The discretized dispersion function G_D lost the property of analyticity in the complex plane a (it has poles on the real axis $\mathrm{Im}\, a = 0$)!

Discretized function $\mathrm{G_D}$

Plots of real and imaginary parts of the function $\mathrm{G_D}$ (with $\frac{\sin \varkappa\Delta\zeta}{\Delta\zeta\varkappa} \to 1$)

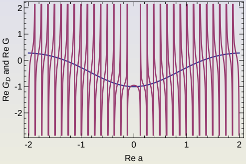


$\mathrm{G_D}$ agrees very well with G in the half-plane $\mathrm{Im}\, a > 0$, however for $\mathrm{Im}\, a < 0$, $\mathrm{G_D} \neq \mathrm{G}$ (we cut out a narrow stripe in the vicinity $\mathrm{Im}\, a \approx 0$). Hence we cannot find Landau damped modes in the discretized system, but the unstable ones should be the same as for the continuous one!

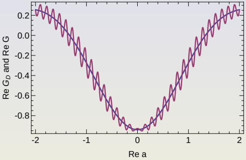
Discretized function G_D for $\mathrm{Im}\, a \approx 0$.

Discretized G_D function (magenta) and continuous function G (blue) for small $\mathrm{Im}\, a$. The function G_D is computed for $p_{min} = -4$, $p_{max} = 4$ and $\Delta p = 0.125$.

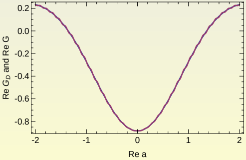
$\mathrm{Im}\, a = 0$



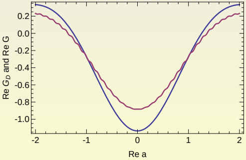
$\mathrm{Im}\, a = 0.05$



$\mathrm{Im}\, a = 0.1$

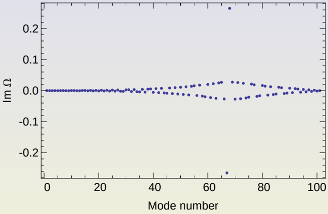


$\mathrm{Im}\, a = -0.1$



Eigenvalues for the discretized dispersion function

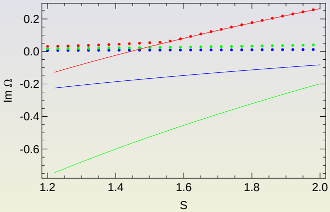
Imaginary parts of Ω for eigenmodes for $S = 2$ $\varkappa = 4\pi/l = \pi/3$ for $p_{\min} = -4$, $p_{\max} = 4$ and $\Delta p = (p_{\max} - p_{\min})/100$ and $\Delta\zeta = L/100$.

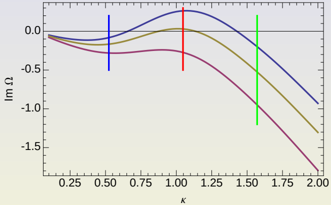


The roots always come in pairs, ω and $-\omega$. In addition to the physical mode with $\mathrm{Im}\,\Omega \approx 0.3$, there are many spurious modes, but they all have small values of $\mathrm{Im}\,\Omega$.

Eigenvalues for the discretized function

Imaginary parts of Ω versus S for eigenmodes with $\varkappa = 2\pi/l = \pi/6$ (blue), $\varkappa = 4\pi/l = \pi/3$ (red) and $\varkappa = 8\pi/l = 2\pi/6$ (green) for $p_{\min} = -4$, $p_{\max} = 4$ and $\Delta p = (p_{\max} - p_{\min})/100$ and $\Delta\zeta = L/100$. Solid lines—continuous model, dots—the discretized one.

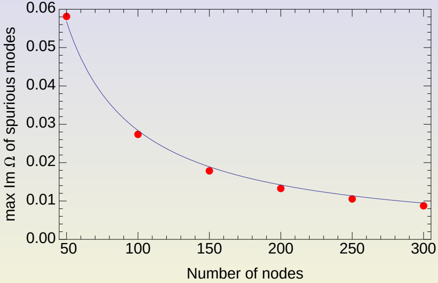




Conclusion: the discretized model correctly finds unstable modes. It fails in predicting the damping rate of the stable modes, but we do not care about those.

Spurious modes vs number of mesh nodes

The max values of $\mathrm{Im}\,\Omega$ for spurious modes decreases (linearly) with the mesh size.



Horizontal axis shows the number of nodes in ζ (equal to the number of nodes in p). The line is a hyperbola fit.

Eigenmodes of discretized system

Let us pretend that we do not know that eigenmodes $\propto e^{i\varkappa_r \zeta_i}$ and try to solve the original eigenproblem. If $\mathcal{F}$ is a vector of $f_{i,j}$ then we have the eigenvalue problem

$$\lambda\mathcal{F} = A \cdot \mathcal{F}$$

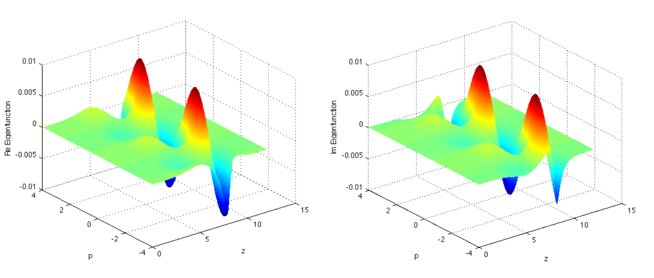
The length of the vector $\mathcal{F}$ is $mn \sim n^2$ (if $m \sim n$), the size of the matrix A is $m^2n^2 \sim n^4$. *This matrix is not sparse*. One would like to be able to solve $m \sim n \sim 200 - 400$, then $m^2n^2 \sim 1.6 \times 10^9 - 2.6 \times 10^{10}$. Standard solvers cannot do such large matrices.

Using ARPACK

ARPACK is a software package that can find several eigenvalues of a large sparse matrix with the largest values of real part. It is based upon “an algorithmic variant of the Arnoldi process called the Implicitly Restarted Arnoldi Method (IRAM)”. One of the options of the ARPACK is that the user provides a subroutine which computes $A \cdot \mathcal{F}$, instead of A (then the matrix can be not sparse). The ARPACK is also implemented as `eigs` function in Matlab.

Solving complete discretized system with eigs

I computed the eigenvalues and eigenfunctions with Matlab eigs function. I used the matrix 400x400 (40 min of calculation with a good initial value), smaller matrices run faster. The largest 'lr' eigenvalue found (out of 2.56×10^{10}) is $\Omega = 0.2634 + 1.4066i$, compared with the continuous model $0.2633 + 1.4069i$.



Real and imaginary part of the eigenfunction with $\varkappa = 4\pi/l = \pi/3$ and $S = 2$.

Bunched beam

For a bunched beam one has to solve first the Haissinski equation and find an equilibrium for the beam.

$$f_0(\eta, z) = \frac{1}{\sigma_\eta} F_0\left(\frac{\eta}{\sigma_\eta}\right) N(z), \qquad K_0(z) \neq 0$$

Dimensionless variables,

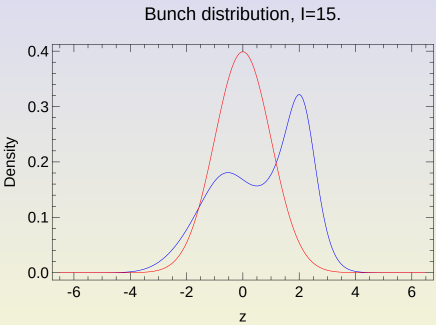
$$\zeta = \frac{z}{\sigma_z}, \quad p = -\frac{\eta}{\sigma_\eta}, \quad \tau = t\omega_{s0}, \quad I = \frac{r_e N_b}{2\pi\gamma\nu_{s0}\sigma_\eta} w(0)$$

and

$$\mathcal{F}_0(p, \zeta) = F_0(p) N(\zeta)$$

Solution of Haissinski equation

The broadband resonant impedance with $Q = 1$ and $\omega_0\sigma_z/c = 1$, bunched beam



Solution of the Haissinki equation.

Discretized system of equations, bunched beam

Again, solutions are $\propto e^{\lambda t}$

$$\lambda f_{i,j} = -p_j \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta\zeta} - \hat{K}_{0i} \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta p} - \hat{K}_{1i} \mathcal{F}'_{0,j}$$

and

$$\hat{K}_{1i} = I \Delta p \Delta z \sum_{l=1}^{n} W_{l-i} \sum_{q=1}^{m} f_{l,q}.$$

It turns out that the ARPACK does not scale well for this system (too slow with $n \sim m > 100$).

Using EXPOKIT

There is a software package, EXPOKIT, which computes $\exp(\tau A) \cdot \mathcal{F}$ "using Krylov subspace projection techniques". It also has an option of supplying to it a subroutine that computes the product $A \cdot \mathcal{F}$ instead of A itself.

If the system is unstable, then for τ large enough, only the eigenvalue with the largest real part survives. Starting from a random $\mathcal{F}_{\text{init}}$

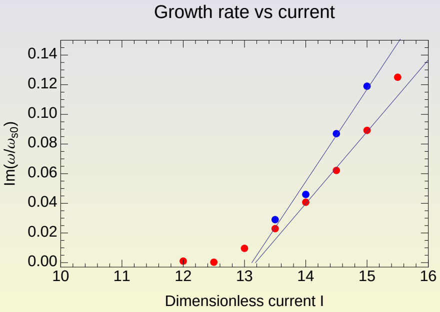
$$\mathcal{F}_{\text{fin}} = \exp(\tau A) \cdot \mathcal{F}_{\text{init}}$$

gives $\mathcal{F}_{\text{fin}}$ which is close to the eigenvalue with the max real part. The growth rate can then be found using norm $\|...\|$

$$e^{\tau \operatorname{Re} \lambda} \approx \| \exp(\tau A) \cdot \mathcal{F}_{\text{fin}} \| / \| \mathcal{F}_{\text{fin}} \|$$

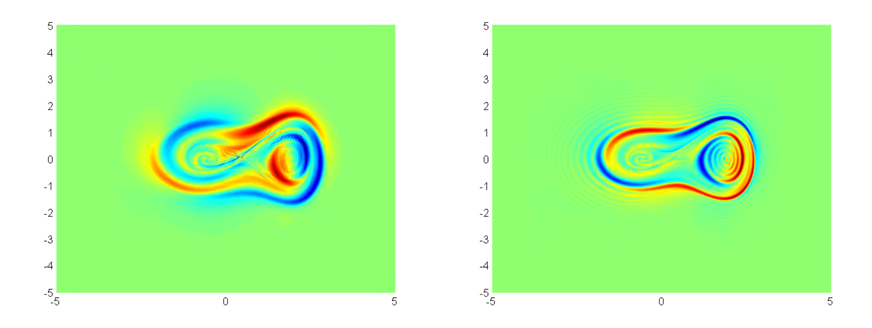
Threshold of the instability

Growth rate versus current for a bunched beam with resonant broadband impedance ($Q = 1$ and $\omega_0\sigma_z/c = 1$): red—from our PRST-AB paper, blue—new Matlab one (with EXPOKIT). Mesh size up to 400×400.



Unstable Eigenmode

Left: phase plot of an unstable eigenmode for $\mathrm{I} = 14.5$, mesh: 250×250; right $\mathrm{I} = 13.5$, mesh: 400×400.



Conclusions

- Solving LVE is usually sufficient for stability analysis on the design phase, where the main issue is finding the threshold of the instability and the growth rate.
- LVE fails to predict Landau damped modes in the system, but correctly finds unstable perturbations.
- Numerical solution of LVE requires finding of the eigenvalues with the largest real part of a huge system of linear equations. ARPACK and EXPOKIT seem like good tools for this problem. The algorithm can be implemented in a simple code. ARPACK has a parallelized implementation.
- Further optimization and extension of the method is envisioned: better discretization algorithms; various wakefields, including RW, CSR, etc. Benchmarking of the method.