A New Two Particle Model for Study of Effects of Space-Charge Force on Beam Instabilities*.

Yong Ho Chin (KEK), Alex Chao (SLAC) and Mike Blaskiewicz (BNL)

J-PARC Accelerator Seminar
June 5, 2015

*Paper Submitted to PRST-AB for Publication.
Outline

- Motivations of This Work
- Original Two Particle Model
- A New Two Particle Model with Space-Charge
- Findings of This Work
- Introduction of Some Other People’s Works
  - Blaskievicz’s Airbag Model for Strong Head-Tail Instability with Space-Charge
  - Kornilov’s FAIR simulations and some measurements.
No beam instability has been observed at RCS.

It is generally believed that a beam at RCS is stabilized by a large incoherent tune spread (Landau damping) due to non-linearity of the space-charge force.

Shobuda-san even proposed to shorted a bunch at RCS to increase the space-charge force to achieve a stronger damping of a beam, though it sounds contrary to common belief (a longer bunch is more stable).

Is the space-charge force really a “magic cure”? 
Mysterious Simulation/Analytic Results

- During HB2014 Workshop, Kornilov and Blaskiewicz reported mysterious simulation and analytical results for beam instabilities with space-charge force.

**FIG. 10.** Growth rates of the most unstable head-tail modes obtained in simulations for a Gaussian bunch for three different head-tail phases $\chi = \xi \tau_b$ in a dependence on the space-charge parameter. The mode index $k$ is given for each data point with the corresponding color.

**FIG. 1.** Largest value of $\text{Im}(\Delta Q_s/Q_s)$ as a function of $\Delta Q_{sc}/Q_s$ and $m^{\text{max}}$ using matrix element (12). The value of $W$ is twice the size needed to produce instability with $\Delta Q_{sc} = 0$: $m^{\text{max}} = 1$, solid line; $m^{\text{max}} = 5$, squares; $m^{\text{max}} = 10$, circles.

V. Kornilov and O. Boine-Frankenheim
PRST-AB, 13, 114201 (2010)

M. Blaskiewicz
PRST-AB, 1, 044201 (1998)
Beam Instabilities with Space-Charge

- Many simulation results generally indicate that beam instability can be damped by a weak space-charge force, but the beam becomes unstable again when the space charge force is further increased.
- If the damping of beam instabilities is caused by the betatron tune spread (Landau damping) due to the non-linearity of the space-charge force,
  - A stronger space-charge force should be more effective in damping of beam instabilities.
- Why do many simulation results show the contrary?
- This mystery has not been solved for ~20 years.
After the working session at HB2014, Chao has invited me to collaborate on study for effects of space-charge force on beam instabilities by modifying his famous two particle model for a strong head-tail instability.

That was a fascinating idea.

We may be able to solve the mystery by using a simple model and mathematics for this complicated phenomenon.

We found later though that his proposed new two particle model did not work (a pity).

So, it turned out that the crux of the problem is to find a suitable new two particle model which is

A simple expansion of the original two particle model

Still analytically solvable.
According to this book,

- Westerners think that the World is simple and steady.
  - It is ruled by simple laws of nature and can be described by simple models.
  - They value principles.

- Asians think that the World is complicated and rapidly changing.
  - It is too complicated even to describe.
  - There is no law of nature, since such a law is also changing all the time.
  - They value practicality.

That is why westerners succeeded in creating and developing science called physics, while We Asians failed.

Let's do it in their way at this time to see if it works.
Let us first review the premise and treatment of Chao's original two particle model.

- Two macro-particles executing synchrotron and betatron oscillations.
- Their synchrotron oscillations have equal amplitude, but opposite phases.

\[ 0 < \frac{s}{c} < T_s/2 \]

\[ T_s/2 < \frac{s}{c} < T_s, \]

\[ \gamma''_1 + \left( \frac{\omega_\beta}{c} \right)^2 \gamma_1 = \frac{N T_0 W_0}{2 \gamma c} \gamma_2 \]

\[ \gamma''_2 + \left( \frac{\omega_\beta}{c} \right)^2 \gamma_2 = 0 \]

\( Y_1 \approx Y_2 \)
Total Matrix for Full Synchrotron Period

\[
\begin{bmatrix}
\tilde{y}_1 \\
\tilde{y}_2
\end{bmatrix}_{s=ct_s/2} = e^{-i\omega_\beta T_s/2} \begin{bmatrix} 1 & i\gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix}
\tilde{y}_1 \\
\tilde{y}_2
\end{bmatrix}_{s=0}
\text{ for } 0 < \frac{s}{c} < \frac{T_s}{2}
\]

where

\[
\gamma = \frac{\pi N r_0 W_0 c^2}{4\gamma C \omega_\beta \omega_s}
\]

Dimensionless Wake Field Strength Parameter

Total Matrix

\[
\begin{bmatrix}
\tilde{y}_1 \\
\tilde{y}_2
\end{bmatrix}_{s=ct_s} = e^{-i\omega_\beta T_s} \begin{bmatrix} 1 & 0 \\ i\gamma & 1 \end{bmatrix} \begin{bmatrix}
\tilde{y}_1 \\
\tilde{y}_2
\end{bmatrix}_{s=0}
\]

\[
e^{-i\omega_\beta T_s} \begin{bmatrix} 1 - \gamma^2 & i\gamma \\ i\gamma & 1 \end{bmatrix} \begin{bmatrix}
\tilde{y}_1 \\
\tilde{y}_2
\end{bmatrix}_{s=0}.
\]
Eigenvalues and Growth Rate

The two eigenvalues are

\[ \lambda = \begin{cases} 
1 - \frac{\gamma^2}{2} \pm \sqrt{\frac{\gamma^2}{2} \cdot \left(\frac{\gamma^2}{2} - 2\right)} & \text{if } \gamma^2 \geq 4 \\
1 - \frac{\gamma^2}{2} \pm i \sqrt{\frac{\gamma^2}{2} \cdot \left(2 - \frac{\gamma^2}{2}\right)} & \text{if } \gamma^2 \leq 4
\end{cases} \]

If \( \gamma^2 \geq 4 \), one of the solutions is unstable.

\[ \lambda = 1 - \frac{\gamma^2}{2} - \sqrt{\frac{\gamma^2}{2} \cdot \left(\frac{\gamma^2}{2} - 2\right)} \leq -1 \]

At the threshold value of \( \gamma^2 = 4 \), the eigenvalue \( \lambda \) becomes exactly minus one (\( \lambda = -1 \) or \( \lambda = e^{\pm i\pi} \)).

The frequency shift is

\[ (\omega_\beta + \Delta \omega_\beta) \cdot T_s = \omega_\beta \cdot T_s \pm \pi \quad \Rightarrow \quad \Delta \omega_\beta = \pm \frac{\pi}{T_s} = \pm \frac{\omega_s}{2}. \]
Transverse Mode-Coupling Instability

- It implies that the strong head-tail instability occurs by the mode coupling between the two solutions when the difference of their phase advances over one synchrotron period becomes exactly $2\pi$.

- The growth rate $g$, when $\gamma^2 \geq 4$, is obtained by equating

  $|\lambda| = e^{gT_s} = \sqrt{\frac{\gamma^2}{2} \cdot \left(\frac{\gamma^2}{2} - 2\right)} + \frac{\gamma^2}{2} - 1$.

- The formula for the growth rate:

  $g = \frac{1}{T_s} \log \left\{ \sqrt{\frac{\gamma^2}{2} \cdot \left(\frac{\gamma^2}{2} - 2\right)} + \frac{\gamma^2}{2} - 1 \right\}$.
Instability Mechanism

\[ \sqrt{\frac{\gamma^2}{2} \cdot \left(2 - \frac{\gamma^2}{2}\right)} \]

Phase advance of head-tail mode after one synchrotron period

Y=0

Y=1

1 - \frac{\gamma^2}{2}

Y=2

Y>2
New Two Particle Model with Space Charge

- **Two approximations:**
  - **Linear Model**
    - The space-charge force is linear in the relative distance between the two particles.
  - **Continuous Interaction Model**
    - The two particles interact continuously and coherently with a space charge force in the transverse plane.

\[
K = N r_0 \alpha^2 \beta^2 \gamma^3 C
\]

\[
W = \frac{N r_0 W_0}{2 \gamma C}
\]

For \(0 < \frac{s}{c} < \frac{T_s}{2}\)

\[
y''_1 + \left(\frac{\omega \beta}{c}\right)^2 y_1 = K(y_1 - y_2) + W y_2
\]

\[
y''_2 + \left(\frac{\omega \beta}{c}\right)^2 y_2 = K(y_2 - y_1)
\]

\[
1/\gamma
\]

\[
\tau
\]

2015/6/6

Chin, Chao and Blaskiewicz
The two coupled equations of motion can be solved using the eigenvalue/eigenvector technique.

The stability diagram for the weak space-charge case ($r = K/W \leq 1$).

Unstable regions are shown shaded.

Absolutely stable regardless of $W$
The stability diagram for the strong space-charge case \((r=K/W \geq 1)\). The stability diagram for the weak space-charge case \((r=K/W \leq 1)\) is also plotted for completion. Unstable regions are shown shaded.
Procedure to Calculate Growth rate

For given $\gamma$ (the dimensionless wake field parameter) and $\Delta \nu_{sc}/\nu_s$ (the dimensionless space-charge parameter),

$$r = \frac{K}{W} = \frac{\pi}{2\gamma} \left( \frac{\Delta \nu_{sc}}{\nu_s} \right)$$

$\therefore r \leq 1$

\[ y = 2\sqrt{r(1-r)} \]

$\tanh^2 \left( \frac{\gamma}{2} y \right) \leq y^2$

Yes

Stable

No

Unstable

\[ r^2 = 2 \cdot \frac{1-y^2}{y^2} \cdot \frac{\tanh^2 \left( \frac{\gamma}{2} y \right)}{1-\tanh^2 \left( \frac{\gamma}{2} y \right)} \]

$\therefore r \geq 1$

\[ y = 2\sqrt{r(r-1)} \]

$\tan^2 \left( \frac{\gamma}{2} y \right) \leq y^2$

Yes

Stable

No

Unstable

\[ \frac{r^2}{2} = 2 \cdot \frac{1+y^2}{y^2} \cdot \frac{\tan^2 \left( \frac{\gamma}{2} y \right)}{1+\tan^2 \left( \frac{\gamma}{2} y \right)} \]

Growth rate: $g = \frac{1}{T_s} \log \left\{ \sqrt{\frac{r^2}{2}} \cdot \left( \frac{r^2}{2} - 2 \right) + \frac{r^2}{2} - 1 \right\}$
Contour Plots for Growth Rate

These figures are all universal!

Flat contour plot for the growth factor $g \times T_s$ as a function of $\Upsilon$ and $\Delta \nu_s / \nu_s$.

3-dimensional contour plot for the growth factor $g \times T_s$ as a function of $\Upsilon$ and $\Delta \nu_s / \nu_s$. 
Instability Mechanism

\[ Y, W = 0 \]

\[ 1 - \frac{\Gamma^2}{2} \]

\[ \Gamma \propto \sqrt{K(K - W)} \]

\[ \frac{\Delta \nu_{sc}}{2} \]

\[ \frac{\Delta \nu_{sc}}{2} \]

Phase advance of head-tail mode after one synchrotron period

\[ \Gamma = 2 \]

\[ \Gamma > 2 \]
Growth Rate as a Function of Space-Charge Tune Shift

- $\gamma = 4$ case.

- It shows that the space-charge force loses its damping effect when it is too strong.

- It qualitatively reproduces typical behaviors shown in many theoretical and simulation results.
The Mystery Solved?: Linear Coherent Kicks

- The present two particle model has no tune spread effect, since the space-charge force is linearized in the transverse position, and the two particles have identical betatron tunes.

- The damping of beam instabilities with a weak space-charge force is caused by pure coherent kicks of the space-charge force in a way to partially neutralize the coherent wake field kicks.

- The loss of the damping effect with a strong space-charge force is due to an unfavorable combination between the coherent space-charge kicks and the coherent wake field kicks.

- The present model suggests that the main damping mechanism of beam instabilities with a weak space-charge force (as observed in many simulations) is linear coherent space-charge kicks, not the Landau damping due to the non-linearity of the space-charge force (Wow!).
How Landau Damping Works?

- **Non-linear Magnet Case:**
  
  If particles reach a large amplitude, they feel the non-linear field from a magnet.

- **Beam Created Non-linear Field Case:**
  
  If particles oscillate in phase on the transverse plane, they hardly feel the non-linear field from a beam even when they have large oscillation amplitude.
Biased Perception

- The damping of beam instabilities by linear coherent kicks is unusual?
  - No. When we damp beam instabilities externally, we use
    - Non-linear magnets such as octupoles.
      - For Landau damping by an incoherent tune spread.
    - Feedback system.
      - For linear (in the transverse displacement of a beam from the center orbit) coherent kicks to a beam.

- However, when we study on damping of beam instabilities by a beam itself, we tend to think only Landau damping as a damping mechanism of space-charge force (because of a large tune spread).
Two Cases of Absolutely Stable Coupled Motions

As the space-charge force increases, Eqs. of motion approach to those for two pendulums connected with a spring.

\[ y_1'' + \left( \frac{\omega \beta}{c} \right)^2 y_1 = K(y_1 - y_2) \]
\[ y_2'' + \left( \frac{\omega \beta}{c} \right)^2 y_2 = K(y_2 - y_1) \]

Another absolutely stable motions.

\[ y_1'' + \left[ \left( \frac{\omega \beta}{c} \right)^2 - \frac{W}{2} \right] y_1 = \frac{W}{2} y_2 \]
\[ y_2'' + \left[ \left( \frac{\omega \beta}{c} \right)^2 - \frac{W}{2} \right] y_2 = -\frac{W}{2} y_1 \]
Why is the pure space-charge oscillation stable?

- Because there is no energy transfer or flow into the transverse oscillation externally or from the longitudinal motion of a beam.
  - In case of transverse head-tail beam instability, even if only transverse mode is excited in a structure, it has a longitudinal impedance and the head-tail mode gains energy from the longitudinal energy loss of the beam.

- Most of cases, a space-charge tune shift is far larger than a synchrotron tune.
  - Then, we wonder why many space-charge studies are more or less concentrated on cases when they are comparable.

- Let us take a look at their works to get some clues.