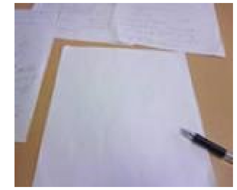
Crystalline Symmetry and Topology

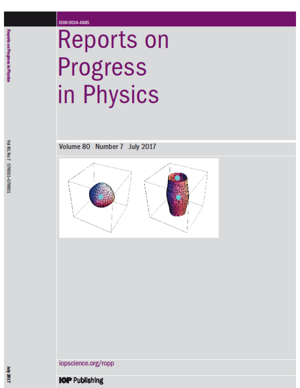
YITP, Kyoto University
Masatoshi Sato

In collaboration with

- Ken Shiozaki (YITP)
- Kiyonori Gomi (Shinshu University)
- Nobuyuki Okuma (YITP)



- Ai Yamakage (Nagoya University)
- Shingo Kobayashi (Nagoya University)
- Yukio Tanaka (Nagoya University)



A review paper on topological SCs with Yoichi Ando

MS, Ando, Rep. Prog. Phys. 80, 076501 (17)

Outline

1. Topological (crystalline) insulators/superconductors

2. K-group classification

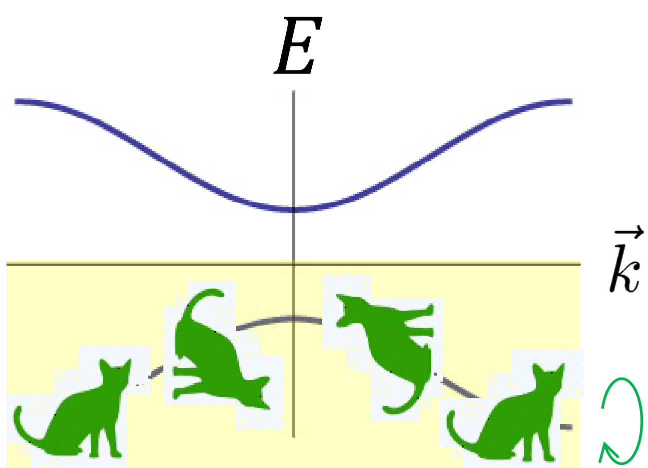
3. Band-theory and Atiyah-Hirzebruch spectral sequence

Introduction

The idea of topological insulator/superconductor (TI/TSC) has been successfully established with many experimental supports for surface states

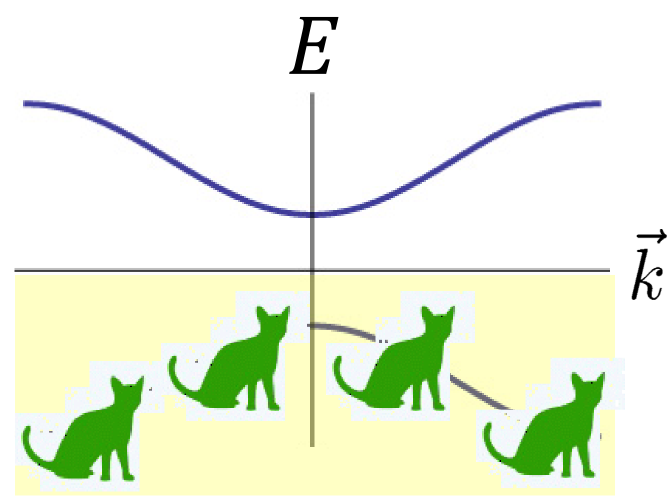
TI/TSC = Non-trivial topological # of occupied state

TI/TSC



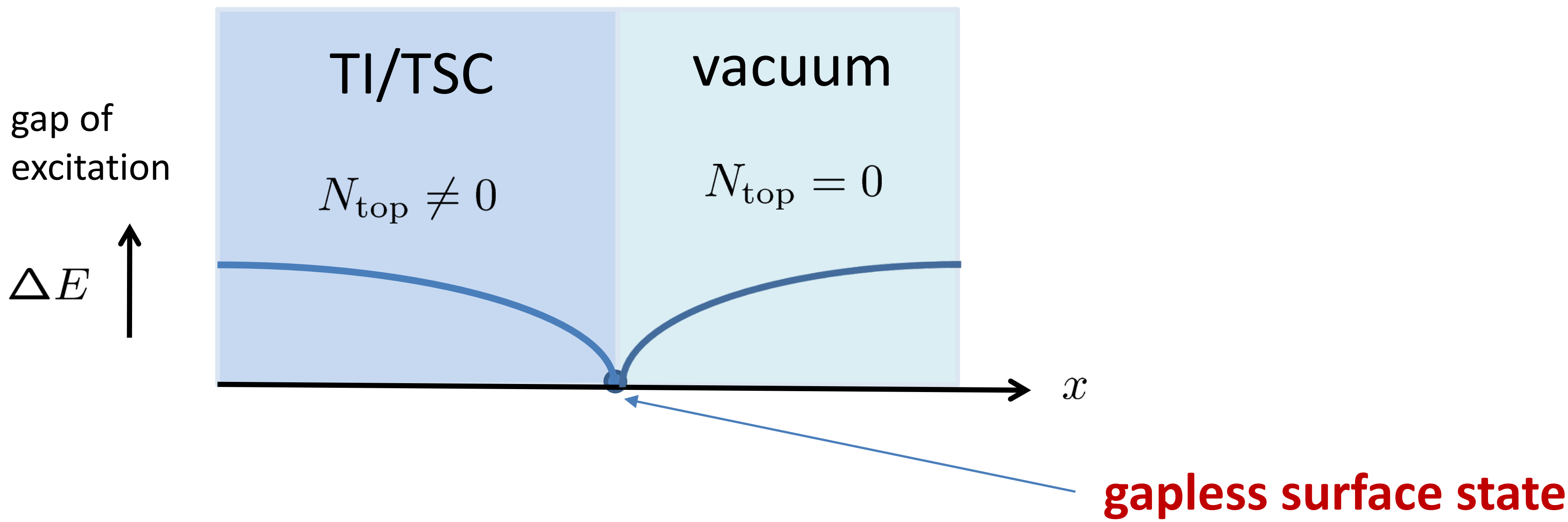
$N_{\text{top}} \neq 0$

Ordinary insulator/SC



$N_{\text{top}} = 0$

The non-trivial topological structure predicts the existence of gapless surface states

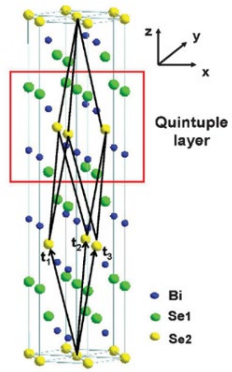


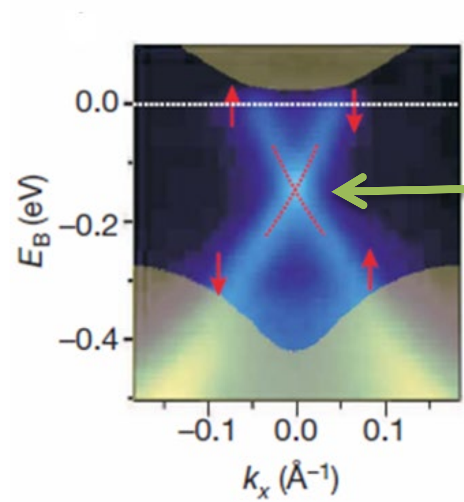
TIs/TSCs have gapless boundary states ensured by bulk topological numbers

bulk-boundary correspondence

Topological insulators

Bi_2Se_3





conduction band

surface Dirac fermions

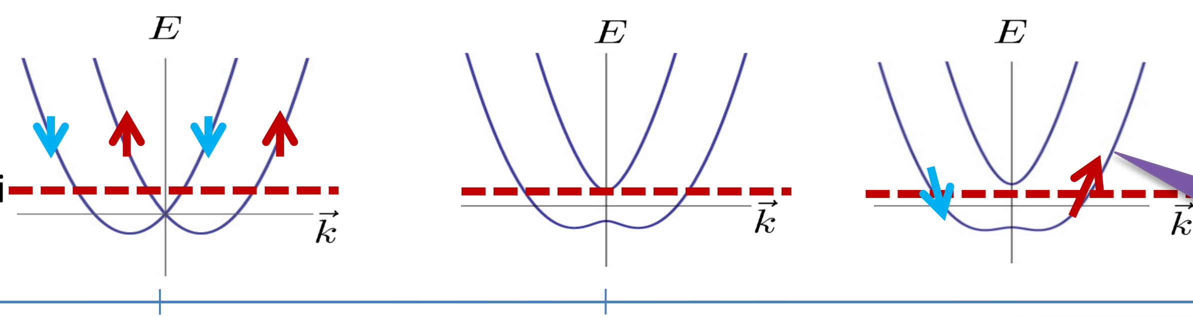
valence band

[Hsieh et al (09)]

Topological superconductors

S-wave SC with Rashba SOC + Zeeman field

[MS-Takahashi-Fujimoto (09), J. Sau et al (10)]



Non spin-degenerate single Fermi surface

Topological SC

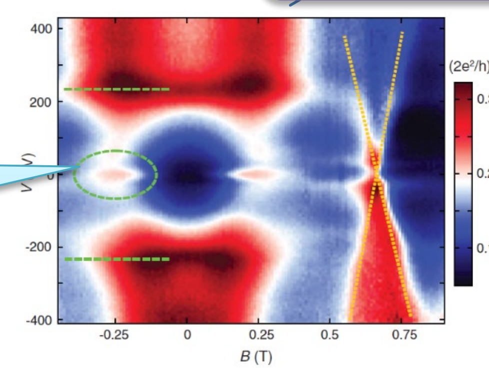
1D Nanowire

[Lutchyn et al (10), Y.O ... 10)]

nanowire

MF

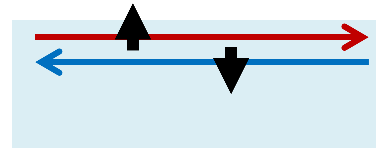
Majorana Fermion



Mourik *et al.*,(12)

Symmetry is very important to obtain top. phases

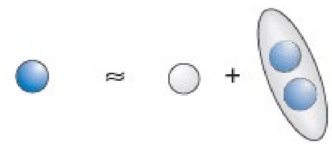
Time-reversal symmetry (TRS)



Kramers pair

- No back scattering
- topologically stable

Particle-hole symmetry (PHS)



Majorana fermion

[Schnyder-Ryu-Furusaki-Ludwig (10)]
[Avron-Seiler-Simon (83)]

	TRS	PHS	CS	d=1	d=2	d=3
A	0	0	0	0	**Z**	0
AIII	0	0	1	**Z**	0	**Z**
AI	1	0	0	0	0	0
BDI	1	1	1	**Z**	0	0
D	0	1	0	Z_2	**Z**	0
DIII	-1	1	1	Z_2	Z_2	**Z**
AII	-1	0	0	0	Z_2	Z_2
CII	-1	-1	1	**2Z**	0	Z_2
C	0	-1	0	0	**2Z**	0
CI	1	-1	1	0	0	**2Z**

IQHS

Majorana nanowire

p+ip chiral p
Sr_2RuO_4, ^{3}He-A

^{3}He-B

$Cu_xBi_2Se_3$

3D TI

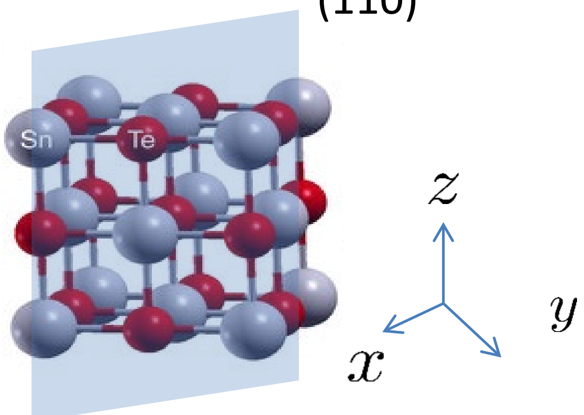
QSH

But this is just a starting point …

Topological Crystalline Insulator [L. Fu (11), Hsieh et al (12)]

Recently, it has been recognized that point group symmetry also provides novel topological surface states

SnTe



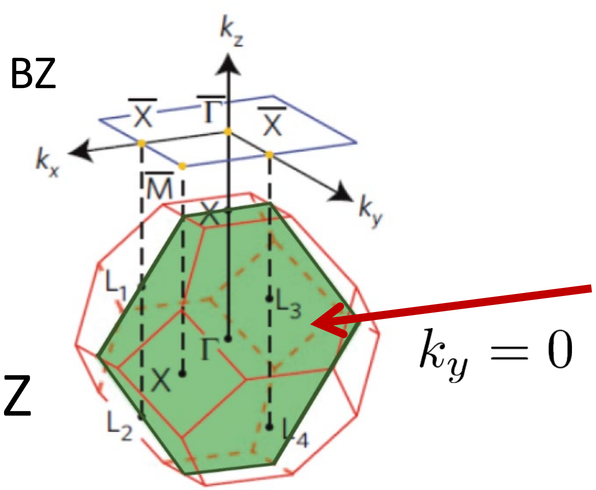
Mirror reflection $\mathcal{M}_{zx}$

$$y \rightarrow -y$$

$$\mathcal{M}_{zx}\mathcal{H}(k_x, k_y, k_z)\mathcal{M}_{zx}^{-1} = \mathcal{H}(k_x, -k_y, k_z)$$

surface BZ

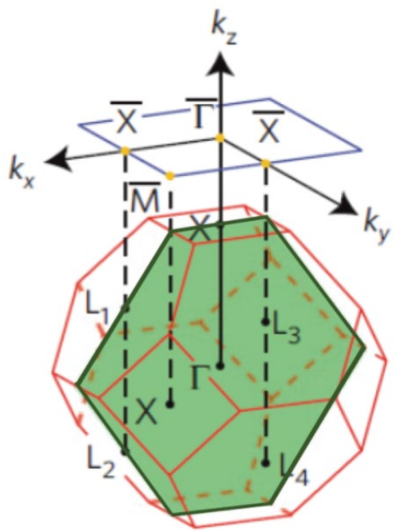
BZ



$$[\mathcal{M}_{zx}, \mathcal{H}(k_x, 0, k_z)] = 0$$

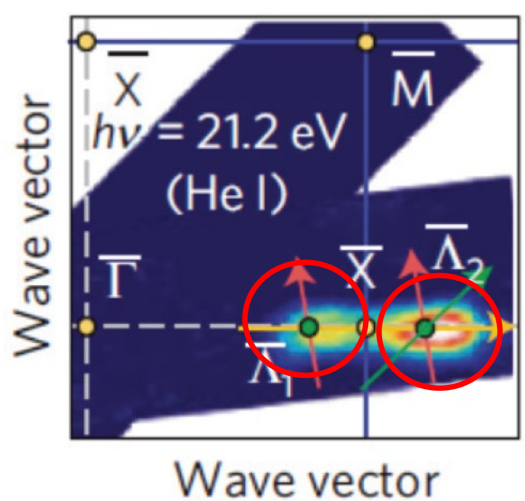
Idea

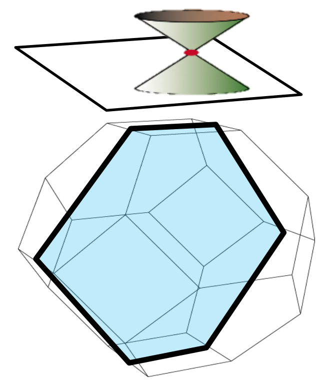
Using the eigen value of mirror operator, ky=0 plane can be separated into two QH states.



[Y. Tanaka et al (12)]

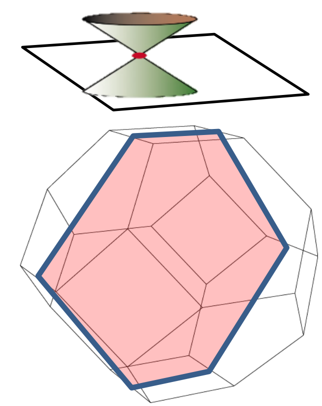
$$[\mathcal{M}_{zx}, \mathcal{H}(k_x, 0, k_z)] = 0$$





$$\nu_{\mathrm{mCh}} = 1$$

$$\mathcal{M}_{zx} = i$$



$$\nu_{\mathrm{mCh}} = -1$$

$$\mathcal{M}_{zx} = -i$$

two Dirac fermions

Questions

- Is it possible to classify such topological phases systematically?
- How many new topological phases can we obtain in the presence of additional symmetry?

To answer these questions, we employ the K-theory.

Shiozaki-MS, Phys. Rev. B90, 166114 (2014).
Shiozaki-MS-Gomi, Phys. Rev. B91, 155120 (2015).
Shiozaki-MS-Gomi, Phys. Rev. B93, 195413 (2016).
Shiozaki-MS-Gomi, Phys. Rev. B95, 235425 (2017). (Editor's suggestion)

K-theory classification of topological crystalline materials

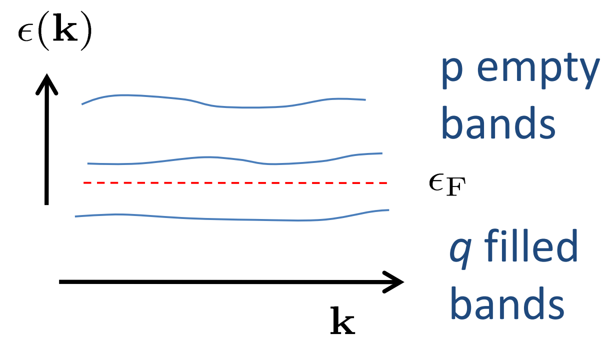
Our setting

In stead of occupied states, we classify flattened Hamiltonians

$$\mathcal{H}(\boldsymbol{k}) = U(\boldsymbol{k})^\dagger \begin{pmatrix} E_1(\boldsymbol{k}) & & \\ & \ddots & \\ & & E_n(\boldsymbol{k}) \end{pmatrix} U(\boldsymbol{k})$$

wave fn.

empty band occupied band

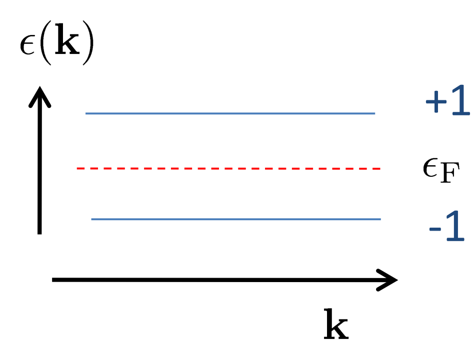


only distinction between empty and occupied band is important

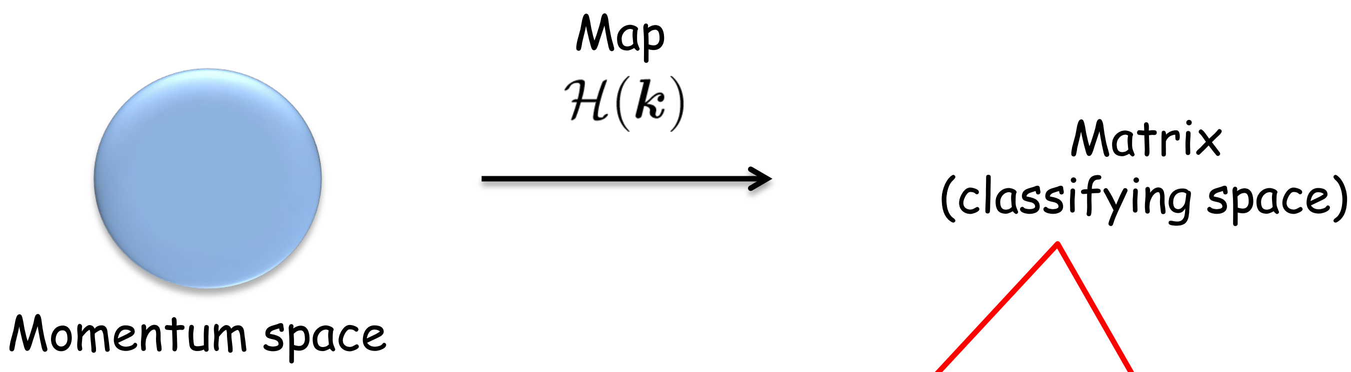
$$\mathcal{H}(\boldsymbol{k}) = U(\boldsymbol{k})^\dagger \begin{pmatrix} \mathbf{1}_{p\times p} & 0 \\ 0 & -\mathbf{1}_{q\times q} \end{pmatrix} U(\boldsymbol{k})$$

$$[\mathcal{H}(\boldsymbol{k})]^2 = 1$$

flattened Hamiltonian



The flattened Hamiltonian defines a map from momentum space to Hilbert space



The problem is how symmetry affects classifying space

If the map defines a non-trivial homotopy, we may have a non-trivial topological phase

Why we use K-theory?

Adding topologically trivial bands makes the classification simpler

[Kitaev(09)]

In addition to simple deformation of Hamiltonians, the K-theory approach allows us to add topologically trivial bands during the deformation of Hamiltonians

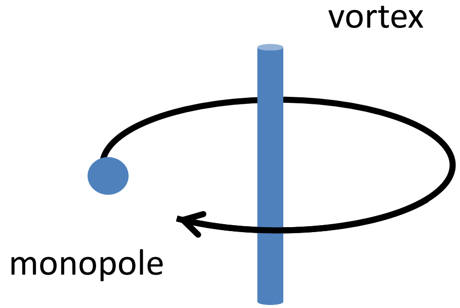
stable equivalence $\mathcal{H}_1 \sim \mathcal{H}_2$

deformable by adding extra trivial bands

The stable-equivalence classes defines topological phases

Importantly, using stable equivalence, we can avoid annoying interference between topological charges

$\mathcal{H}_{\mathrm{vortex+monopole}}$



interference due to AB effect

But if we use stable equivalence …

pair-annihilation of monopoles

trivial band

$$\mathcal{H}_{\mathrm{vortex+monopole}} \sim \begin{pmatrix} \begin{pmatrix} \mathcal{H}_{\mathrm{vortex}} & \\ & \mathcal{H}_{\mathrm{monopole}} \end{pmatrix} & & \\ & \mathcal{H}_{\mathrm{antimonopole}} & \\ & & \mathcal{H}_{\mathrm{monopole}} \end{pmatrix}$$

The vortex and monopole belong to different sectors of Hamiltonian

No interference

The K-theory simplifies the classification of topological phases

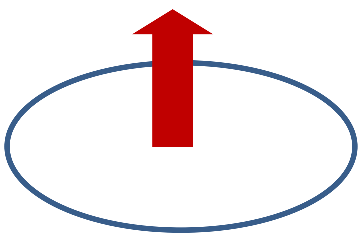
Classification of TCIs and TCSCs: K-theory approach

In general, topological phases can be understood as the existence of topological objects in the momentum space

1 dim top. phase

1dim BZ

= 1dim circle



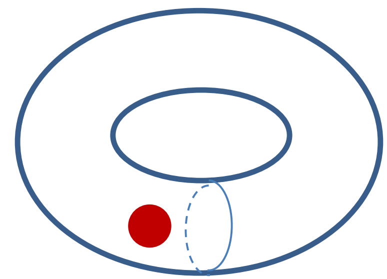
"vortex"

winding number

$$N_w = \oint_{\mathrm{1dBZ}} \frac{dk}{2\pi} \mathrm{tr}\left[\Gamma H^{-1}(\boldsymbol{k}) \partial H(\boldsymbol{k})\right]$$

$$\{\Gamma, H(\boldsymbol{k})\} = 0$$ chiral symmetry

2 dim top. phase



2dim BZ

= 2dim torus

"monopole"

Chern number

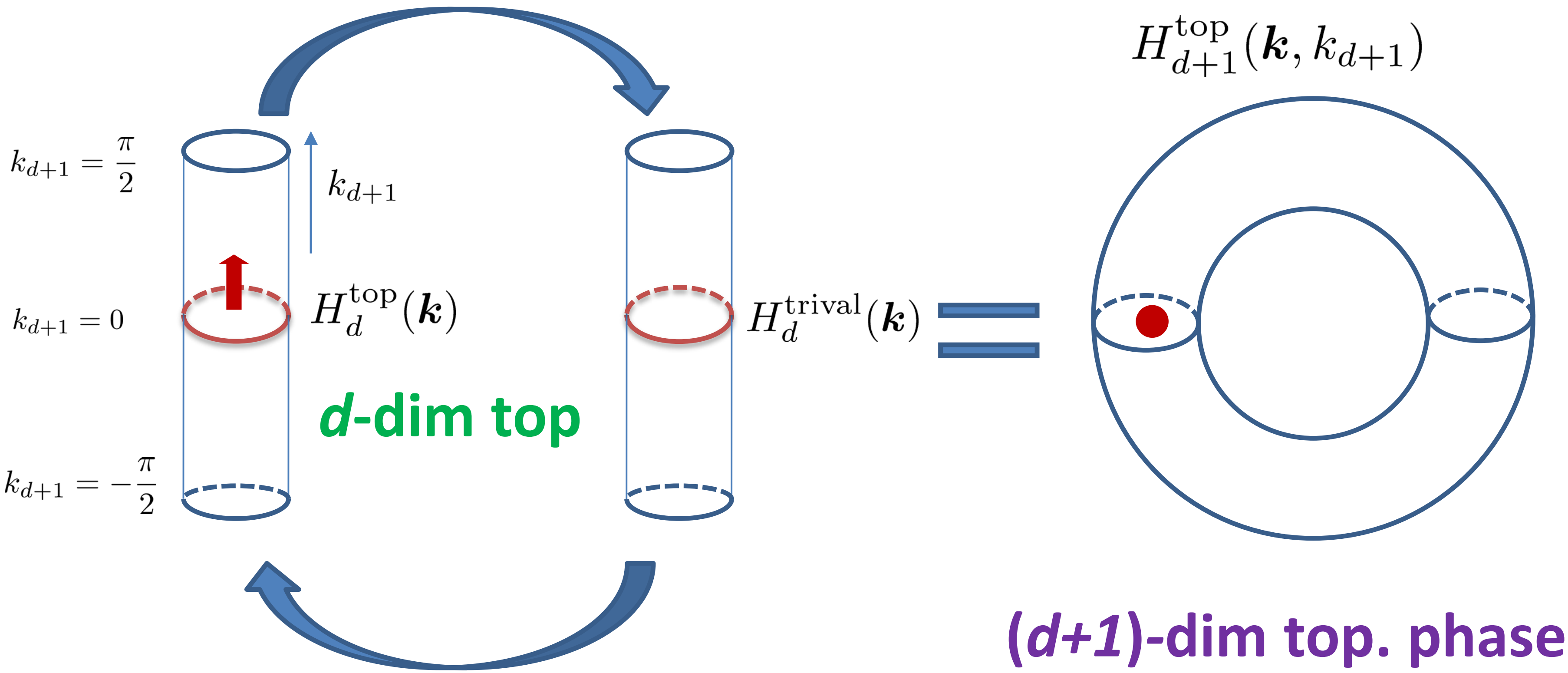
$$Ch_1 = \int_{\mathrm{2dBZ}} \left(\frac{d\boldsymbol{S}}{2\pi}\right) \cdot \left[\boldsymbol{\nabla} \times \mathcal{A}(\boldsymbol{k})\right]$$

$$\mathcal{A}(\boldsymbol{k}) = i \sum_{n \in \mathrm{occ}} \langle u_n(\boldsymbol{k}) | \boldsymbol{\nabla} u_n(\boldsymbol{k}) \rangle$$

In the framework of K-theory, one can increase the dimension of the system systematically

Teo-Kane (10)

$$H_{d+1}(\boldsymbol{k}, k_{d+1}) = \begin{cases} H_d(\boldsymbol{k}) \cos k_{d+1} + \Gamma \sin k_{d+1} & \text{chiral case} \\ (H_d(\boldsymbol{k}) \otimes \sigma_z) \cos k_{d+1} + (1 \otimes \sigma_y) \sin k_{d+1} & \text{non-chiral case} \end{cases}$$



This map keeps the topological number but it shifts the symmetry of the system

class DIII (**TRI SCs**)

$$C H_d(\boldsymbol{k}) C^{-1} = -H_d(-\boldsymbol{k}) \qquad C = \tau_x K \qquad C^2 = 1$$

$$T H_d(\boldsymbol{k}) T^{-1} = H_d(-\boldsymbol{k}) \qquad T = i s_y K \qquad T^2 = -1$$

$$H_{d+1}(\boldsymbol{k}, k_{d+1}) = H_d(\boldsymbol{k}) \cos k_{d+1} + \Gamma \sin k_{d+1} \qquad \Gamma = iTC$$

This term breaks PHS

class AII (**TRI Insulators**)

$$T H_{d+1}(\boldsymbol{k}, k_{d+1}) T^{-1} = H_{d+1}(-\boldsymbol{k}, -k_{d+1})$$

hierarchy of top #

	TRS	PHS	CS	d=1	d=2	d=3
AI	1	0	0	0	0	0
BDI	1	1	1	**Z**	0	0
D	0	1	0	**Z_2**	**Z**	0
DIII	-1	1	1	**Z_2**	**Z_2**	**Z**
AII	-1	0	0	0	**Z_2**	**Z_2**
CII	-1	-1	1	**2Z**	0	**Z_2**
C	0	-1	0	0	**2Z**	0
CI	1	-1	1	0	0	**2Z**

We generalize this idea to systems with additional symmetry

Shiozaki-MS(14)

class DIII (TRI SCs) + **mirror reflection**

$$CH_d(\boldsymbol{k})C^{-1} = -H_d(-\boldsymbol{k}) \qquad C = \tau_x K \qquad C^2 = 1$$

$$TH_d(\boldsymbol{k})T^{-1} = H_d(-\boldsymbol{k}) \qquad T = is_y K \qquad T^2 = -1$$

$$UH_d(\boldsymbol{k})U^{-1} = H_d(-k_1, k_2, \dots, k_d) \qquad U = is_x$$

mirror sym.

$$H_{d+1}(\boldsymbol{k}, k_{d+1}) = H_d(\boldsymbol{k}) \cos k_{d+1} + \underline{\Gamma \sin k_{d+1}} \qquad \Gamma = iTC$$

This term keeps mirror sym.

class AII (TRI Insulators) + **mirror reflection**

$$TH_{d+1}(\boldsymbol{k}, k_{d+1})T^{-1} = H_{d+1}(-\boldsymbol{k}, -k_{d+1})$$

$$UH_{d+1}(\boldsymbol{k})U^{-1} = H_{d+1}(-k_1, k_2, \dots, k_d, k_{d+1})$$

mirror sym.

However, this is not the only possibility ….

The mapped Hamiltonian also has a different additional symmetry

$$(U\Gamma)H_{d+1}(\boldsymbol{k})(U\Gamma)^{-1} = -H_{d+1}(-k_1, k_2, \ldots, k_d, \underline{-k_{d+1}}) \qquad \Gamma = iTC$$

This symmetry flips k_{d+1}

= 2-fold rotation

***d*-dim**

class DIII

$H_d(\boldsymbol{k})$

$T \quad C \quad U$

(*d+1*)-dim

class AII

$H_{d+1}(\boldsymbol{k})$

$T \quad U$

mirror reflection

The same top #

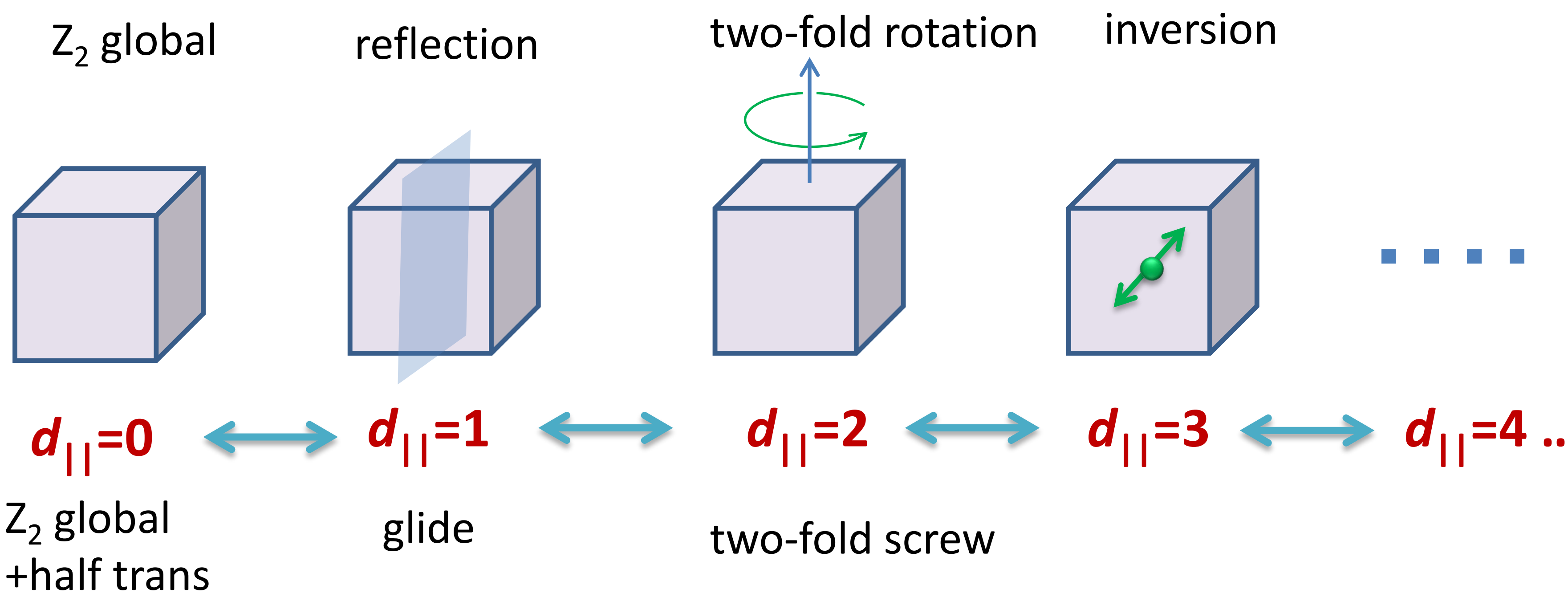
$H_{d+1}(\boldsymbol{k})$

$T \quad U\Gamma$

2-fold rotation

Different additional symmetry but the same topological structure

In this manner, **we can change the number of flipped coordinates $d_{||}$ under the symmetry,** with keeping the topological structure



Using these relations, we have completed the classification of TCIs and TCSCs protected by order-two space (and magnetic space) groups

Extended topological Table

[Shiozaki-MS (14), Shiozaki-MS-Gomi (15)]

A single periodic table with 10 different topological class

[Schnyder et al (08)]

→

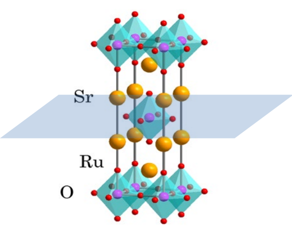
6 periodic tables with 27 classes

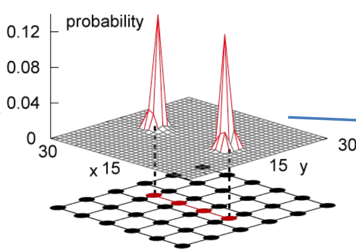
+

6 periodic tables with 10 classes

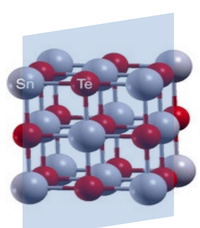
222 class

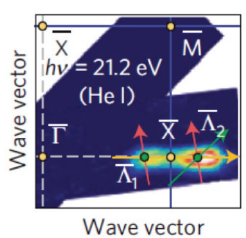
Sr_2RuO_4





SnTe





Symmetry	Class	$\mathcal{C}_q$ or $\mathcal{R}_q$	$\delta=0$	$\delta=1$	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
U	A	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
U_+	AIII	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
U_-	AIII	$\mathcal{C}_1\times\mathcal{C}_1$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$
	AI	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
	BDI	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
	D	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
$U^+_+, U^-_-, U^+_{++}, U^-_{--}$	DIII	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	AII	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CII	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	C	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
U^+_{+-}, U^-_{-+}	BDI	$\mathcal{R}_1\times\mathcal{R}_1$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$	0	0	0	$2\mathbb{Z}\oplus2\mathbb{Z}$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
U^+_{-+}, U^-_{+-}	DIII	$\mathcal{R}_3\times\mathcal{R}_3$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$	0	0	0	$2\mathbb{Z}\oplus2\mathbb{Z}$
U^+_{+-}, U^-_{-+}	CII	$\mathcal{R}_5\times\mathcal{R}_5$	0	$2\mathbb{Z}\oplus2\mathbb{Z}$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$	0	0
U^+_{-+}, U^-_{+-}	CI	$\mathcal{R}_7\times\mathcal{R}_7$	0	0	0	$2\mathbb{Z}\oplus2\mathbb{Z}$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$
	AI	$\mathcal{R}_7$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	$\mathcal{R}_0$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	D	$\mathcal{R}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
$U^+_-, U^-_+, U^+_{--}, U^-_{++}$	DIII	$\mathcal{R}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
	AII	$\mathcal{R}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
	CII	$\mathcal{R}_4$	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	$\mathcal{R}_5$	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	$\mathcal{R}_6$	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
U^+_{-+}, U^-_{+-}	BDI, CII	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
U^+_{+-}, U^-_{-+}	DIII, CI	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

We could take into account order-two space group syms, but they are a merely part of space group syms ….

How to include general space group syms in band topology?

Our answer is to use Atiyah-Hirzebruch Spectral Sequence (AHSS)

Shiozaki-MS-Gomi, arXiv:1802.06694

Band Theory

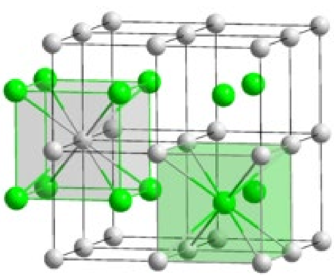
- Energy spectra of electrons in crystals are given in Brillouin zone.
- Each band at the momentum k belongs to an irreducible rep. of crystalline symmetry keeping k (= little group $G_{\boldsymbol{k}}$)

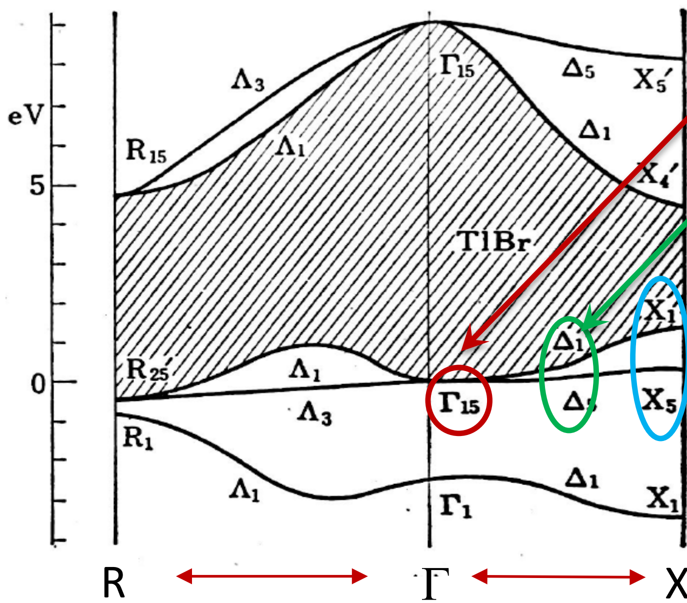
crystal sym. $U_g H(\boldsymbol{k}) U_g^{-1} = H(g\boldsymbol{k})$

bands should be rep of $G_{\boldsymbol{k}}$

For $g \in G_{\boldsymbol{k}}$ (i.e. for $g\boldsymbol{k} = \boldsymbol{k}$) $\Rightarrow$ $[H(\boldsymbol{k}), U_g] = 0$

TlBr O_h^1 (221)

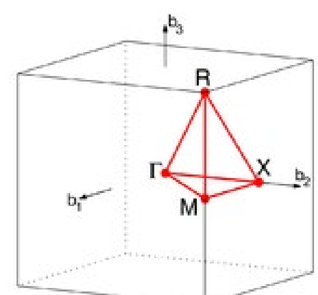




irrep of O_h group (Γ point)

irrep of C_{4v} group (ΓX line)

irrep of D_{4h} group (X point)



These representations in band structures give useful information of topology

Topological Insulator [Fu-Kane (06)]

$$\delta_a = \frac{\mathrm{Pf}[w(\Lambda_a)]}{\sqrt{\mathrm{Det}[w(\Lambda_a)]}}$$ **Z_2 number**

Λ_a TR-invariant momentum

$$\Lambda_a = -\Lambda_a + \boldsymbol{G}$$

occupied state

$$w_{mn}(\Lambda_a) = \langle u_m(\Lambda_a)|T u_n(\Lambda_a)\rangle$$

$\boldsymbol{k} \to -\boldsymbol{k}$

Inversion sym

Representation of inversion sym.

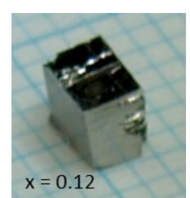
$$\delta_a = \prod_m \xi_m(\Lambda_a)$$

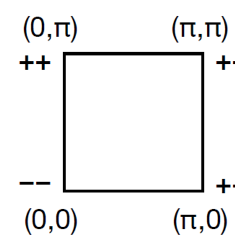
$\xi_m(\Lambda_a)$

Parity of occupied state

- **Non-local**
- **Difficult to evaluate**

$Bi_{1-x}Sb_x$



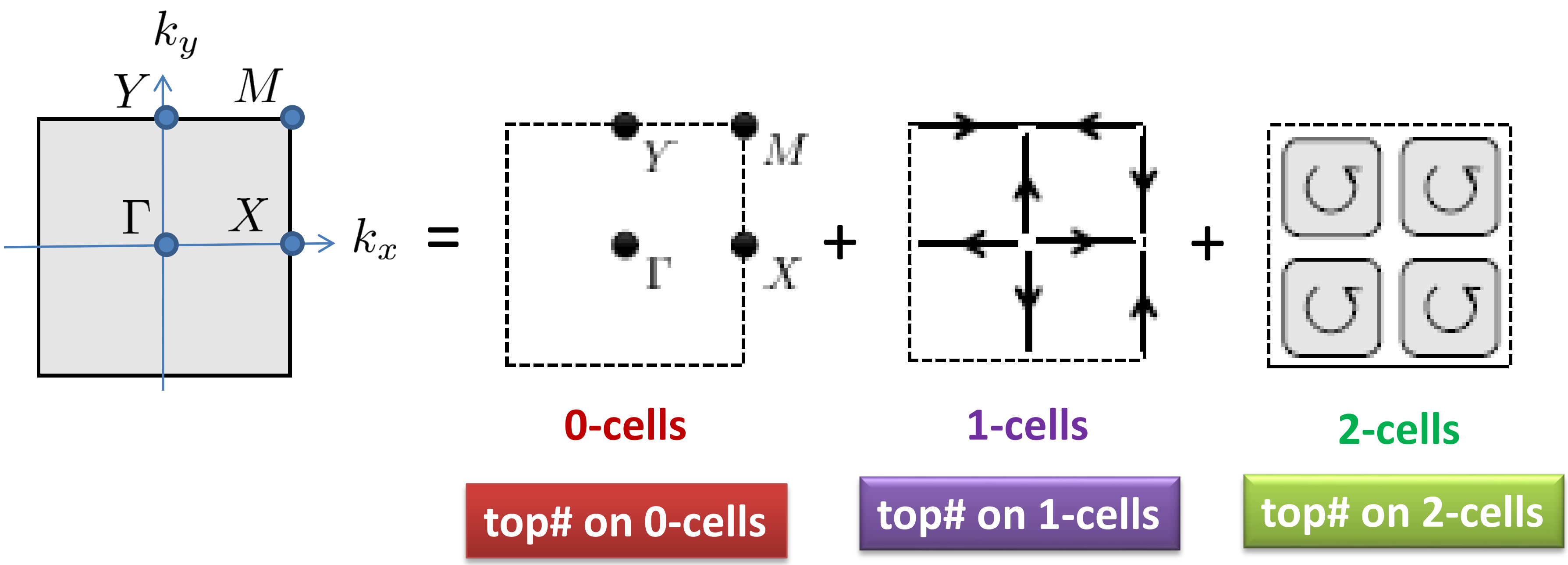


- **Local**
- **Easy to evaluate**

Atiyah-Hirzebruch spectral sequence (AHSS) introduces such band representations naturally in the framework of K-theory.

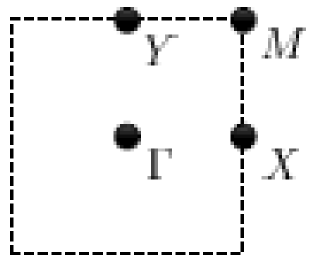
What is AHSS ?

AHSS is a mathematical tool to approximate K-group on the whole BZ by cell-decomposition

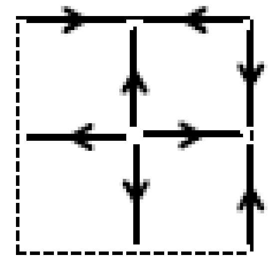


If these top#s on p-cells are consistently extended to the whole BZ, then they should be top#s of K-group on BZ

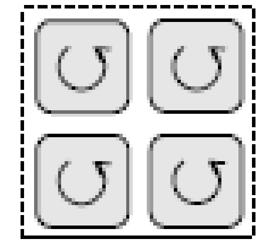
Extension of top# on p-cells to higher-cells can be done iteratively



0-cells



1-cells



2-cells

top# on p-cells

(E1-page)

top# on 0-cells | **top# on 1-cells** | **top# on 2-cells**

Check extension to adjacent (p+1)-cells

top# on p-cells ext to (p+1) cells

(E2-page)

$E_2^{0,*}$ $E_2^{1,*}$ $E_2^{2,*}$

Check extension to (p+2)-cells → E3-page

Check extension to (p+3)-cells → E4-page

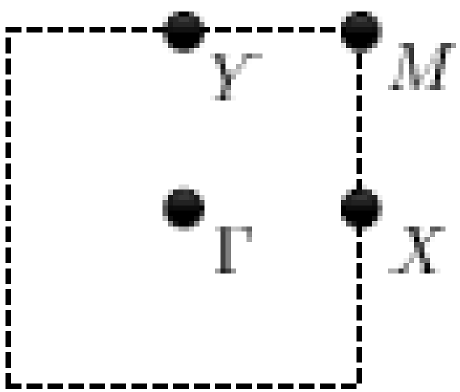
....

Top # defined on whole BZ

$E_\infty^{0,*}$ $E_\infty^{1,*}$ $E_\infty^{2,*}$

Relation to the band theory

For the cell decomposition respecting the crystal symmetry, top#s on p-cells are given by # of irreps of occupied bands under the little group



0-cells = high sym. points

top# on 0-cells

# of irreps of occupied band at high sym. points

Generally, in terms of K-theory, top # on p-cell D^p can be written as

$$E_1^{p,-n} \sim \prod_{D^p;\mathrm{p-cell}} K_{G_{\boldsymbol{k}}}^{-n}(\boldsymbol{k}) \qquad (\boldsymbol{k} \in D^p)$$

# of irreps of occupied band under the little group G_k

Interestingly, using the dimensional shift and bulk-boundary correspondence, we have several different meanings of E1 page

First, by increasing the dimension, one can interpret E1-page as topological insulators

$$E_1^{p,-n} \sim \prod_{D^p;\mathrm{p-cell}} K_{G_{\boldsymbol{k}}}^{-n}(\boldsymbol{k}) \qquad (\boldsymbol{k} \in D^p)$$

Class n
0-dim top # on p-cells

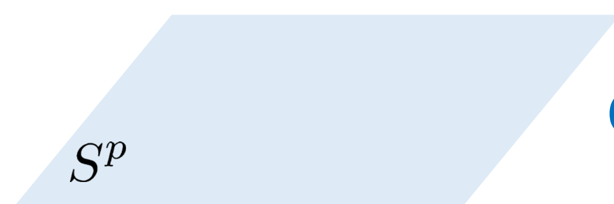
$\Updownarrow$ Hamiltonian map

$$\prod \tilde{K}_{G_{\boldsymbol{k}}}^{-(n-p)}(S^p)$$

$$(S^p = D^p/\partial D^p)$$

Class (n-p)
p-dim top # on p-cells

$E_1^{p,-n}$ =



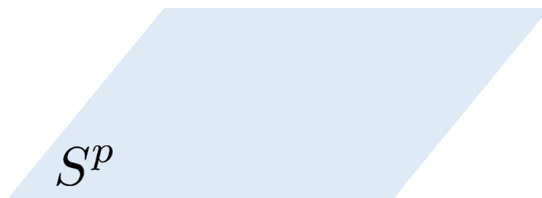
Class (n-p) p-dim top. insulator

Furthermore, combining the dimensional shift and bulk-boundary correspondence, we can also interpret E1-page as gapless states

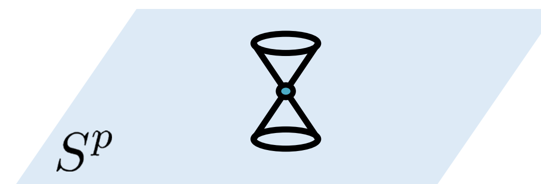
Class (n-p) p-dim TI

Class (n+1-p) p-dim top gapless state

$E_1^{p,-n}$ =



=



Hamiltonian map (dim shift)

Bulk-boundary correspondence

Three different meanings of E1-page

- Irreducible reps. of little group
- Top insulators on p-cells
- Top gapless states on p-cells

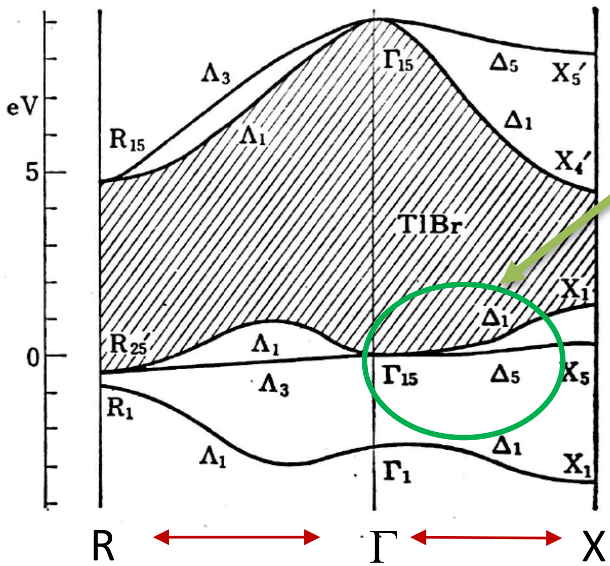
Class (n+1-p) (p+1)-dim TI

How to check the extension

To check the extension, we consider a map d_1 between top # on p-cells and top # on adjacent (p+1)-cells

$$d_1 : E_1^{p,-n} \to E_1^{p+1,-n}$$

Such a map can be obtained naturally if we interpret E1-pages as irreps on p-cells and those on (p+1)-cells



irreps

$$\underbrace{\Gamma_{15}}_{\text{0-cell}} \to \underbrace{\Delta_1 + \Delta_2}_{\text{1-cell}}$$

0-cell (Γ point)

1-cell (ΓX line)

We have relation between irreps. between p- and (p+1) cells

(=compatibility relation)

The compatibility relation defines d_1

The extension from p- to (p+1)-cells can be checked by d_1 with different interpretation of E1-pages

$$d_1 : E_1^{p,-n} \to E_1^{p+1,-n}$$

top. insulator on p-cell **top. gapless state on (p+1)-cell**

Thus, TI on p-cell without gapless state on (p+1)-cells can be obtained as kernel of d_1

$$\boxed{\mathrm{Ker} d_1 = \{\underline{\nu \in E_1^{p,-n}}; d_1\nu = \underline{0}\}}$$

top. insulator on p-cell without gapless state on (p+1)-cell

top# on p-cells ext to (p+1) cells $\Rightarrow$ E_2-page

In this manner, we can check the extension to higher cells and eventually obtain top#s on whole BZ

Using AHSS, we have obtained the complete list of top#s for all 230 space groups (without TRS, PHS)

[Shiozaki-MS-Gomi (18)]

SG	Short	$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}$	$E_2^{3,0}$
1	$P1$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$
2	$P\bar{1}$	$\mathbb{Z}^9$	0	$\mathbb{Z}^3$	$\mathbb{Z}_2$
3	$P2$	$\mathbb{Z}^5$	$\mathbb{Z}^5$	$\mathbb{Z}$	$\mathbb{Z}$
4	$P2_1$	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_2^3$	$\mathbb{Z}$	$\mathbb{Z}$
5	$C2$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}$
6	Pm	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^3$	0
7	Pc	$\mathbb{Z}$	$\mathbb{Z}^2+\mathbb{Z}_2$	$\mathbb{Z}+\mathbb{Z}_2$	0
8	Cm	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	0
9	Cc	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}+\mathbb{Z}_2$	0

SG	Short	$\epsilon(2_{001}, m_{001})$	$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}$	$E_2^{3,0}$
10	$P2/m$	$+_{0,1/2}$	$\mathbb{Z}^{15}$	0	$\mathbb{Z}^3$	0
		$-$	$\mathbb{Z}$	$\mathbb{Z}^8$	$\mathbb{Z}$	0
11	$P2_1/m$	$+_{0,1/2}, -$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
12	$C2/m$	$+_{0,1/2}$	$\mathbb{Z}^{10}$	0	$\mathbb{Z}^2$	0
		$-$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}$	0
13	$P2/c$	$+_{0,1/2}, -$	$\mathbb{Z}^7$	$\mathbb{Z}^2$	$\mathbb{Z}$	0
14	$P2_1/c$	$+_{0,1/2}, -$	$\mathbb{Z}^5$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
15	$C2/c$	$+_{0,1/2}, -$	$\mathbb{Z}^6$	$\mathbb{Z}$	$\mathbb{Z}$	0

SG	Short	$\epsilon(2_{100}, 2_{010})$	$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}$	$E_2^{3,0}$
16	$P222$	$+_0$	$\mathbb{Z}^{13}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^{12}$	0	$\mathbb{Z}$
17	$P222_1$	$+_0, -_{1/2}$	$\mathbb{Z}^5$	$\mathbb{Z}^4+\mathbb{Z}_2$	0	$\mathbb{Z}$
18	$P2_12_12$	$+_0, -_{1/2}$	$\mathbb{Z}^3$	$\mathbb{Z}^2+\mathbb{Z}_2^3$	0	$\mathbb{Z}$
19	$P2_12_12_1$	$+_0, -_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}_4^3$	0	$\mathbb{Z}$
20	$C222_1$	$+_0, -_{1/2}$	$\mathbb{Z}^3$	$\mathbb{Z}^2+\mathbb{Z}_2^2$	0	$\mathbb{Z}$
21	$C222$	$+_0$	$\mathbb{Z}^8$	$\mathbb{Z}+\mathbb{Z}_2$	0	$\mathbb{Z}$
		$-_{1/2}$	$\mathbb{Z}^2$	$\mathbb{Z}^7$	0	$\mathbb{Z}$
22	$F222$	$+_0$	$\mathbb{Z}^7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}_2$	$\mathbb{Z}$
23	$I222$	$+_0$	$\mathbb{Z}^7$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}$
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^6+\mathbb{Z}_2$	0	$\mathbb{Z}$
24	$I2_12_12_1$	$+_0, -_{1/2}$	$\mathbb{Z}^4$	$\mathbb{Z}^3+\mathbb{Z}_2$	0	$\mathbb{Z}$

SG	Short	$\epsilon(m_{100}, m_{010})$	$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}$	$E_2^{3,0}$
25	$Pmm2$	$+_0$	$\mathbb{Z}^9$	$\mathbb{Z}^9$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^4$	0
26	$Pmc2_1$	$+_0$	$\mathbb{Z}^3$	$\mathbb{Z}^3+\mathbb{Z}_2^3$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2+\mathbb{Z}_2$	0
27	$Pcc2$	$+_0$	$\mathbb{Z}^5$	$\mathbb{Z}^5$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2^4$	0
28	$Pma2$	$+_0, -_{1/2}$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}$	0
29	$Pca2_1$	$+_0, -_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_2^2$	$\mathbb{Z}_2$	0
30	$Pna2_1$	$+_0, -_{1/2}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}_2$	0
31	$Pmn2_1$	$+_0, -_{1/2}$	$\mathbb{Z}^2$	$\mathbb{Z}^3+\mathbb{Z}_2$	$\mathbb{Z}$	0
32	$Pba2$	$+_0, -_{1/2}$	$\mathbb{Z}^3$	$\mathbb{Z}^3+\mathbb{Z}_2$	$\mathbb{Z}_2$	0
33	$Pna2_1$	$+_0, -_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_4$	$\mathbb{Z}_2$	0
34	$Pnn2$	$+_0, -_{1/2}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}_2$	0
35	$Cmm2$	$+_0$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	0	0
		$-_{1/2}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	0
36	$Cmc2_1$	$+_0$	$\mathbb{Z}^2$	$\mathbb{Z}^2+\mathbb{Z}_2^2$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^2+\mathbb{Z}_2$	$\mathbb{Z}$	0
37	$Ccc2$	$+_0$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	0	0
		$-_{1/2}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	0
38	$Amm2$	$+_0$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^3$	0
39	$Abm2$	$+_0$	$\mathbb{Z}^4$	$\mathbb{Z}^4+\mathbb{Z}_2$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}+\mathbb{Z}_2^2$	0
40	$Ama2$	$+_0, -_{1/2}$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}$	0
41	$Aba2$	$+_0, -_{1/2}$	$\mathbb{Z}^2$	$\mathbb{Z}^2+\mathbb{Z}_2$	$\mathbb{Z}_2$	0
42	$Fmm2$	$+_0$	$\mathbb{Z}^5$	$\mathbb{Z}^5$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2+\mathbb{Z}_2$	0
43	$Fdd2$	$+_0, -_{1/2}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	0
44	$Imm2$	$+_0$	$\mathbb{Z}^5$	$\mathbb{Z}^5$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}^3+\mathbb{Z}_2$	$\mathbb{Z}^2$	0
45	$Iba2$	$+_0$	$\mathbb{Z}^3$	$\mathbb{Z}^3+\mathbb{Z}_2$	0	0
		$-_{1/2}$	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}_2^2$	0
46	$Ima2$	$+_0$	$\mathbb{Z}^3$	$\mathbb{Z}^3+\mathbb{Z}_2$	0	0
		$-_{1/2}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}$	0

(Cont.)

SG	Short	$(\epsilon(m_{100}, m_{010}), \epsilon(m_{100}, m_{001}), \epsilon(m_{010}, m_{001}))$	$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}$	$E_2^{3,0}$
47	$Pmmm$	$(+,+,+)_0$	$\mathbb{Z}^{27}$	0	0	0
		$(-,-,-)_{1/2}$	$\mathbb{Z}^9$	0	$\mathbb{Z}^6$	0
		$(-,+,+), (+,-,+), (+,+,-)$	$\mathbb{Z}^3$	$\mathbb{Z}^{12}$	0	0
		$(+,-,-), (-,+,-), (-,-,+)$	$\mathbb{Z}^5$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	0
48	$Pnnn$	$(+,+,+)_0, (+,-,-), (-,+,-), (-,-,+)$	$\mathbb{Z}^9$	0	$\mathbb{Z}_2$	0
		$(-,-,-)_{1/2}, (-,+,+), (+,-,+), (+,+,-)$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	0	0
49	$Pccm$	$(+,+,+)_0, (+,-,-)$	$\mathbb{Z}^{14}$	0	$\mathbb{Z}$	0
		$(-,-,-)_{1/2}, (-,+,+)$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}$	0
		$(+,-,+), (+,+,-)$	$\mathbb{Z}$	$\mathbb{Z}^{10}$	0	0
		$(-,+,-), (-,-,+)$	$\mathbb{Z}^5$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	0
50	$Pban$	$(+,+,+)_0, (+,-,-), (-,+,-), (-,-,+)$	$\mathbb{Z}^9$	0	$\mathbb{Z}_2$	0
		$(-,-,-)_{1/2}, (-,+,+), (+,-,+), (+,+,-)$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	0	0
51	$Pmma$	$(+,+,+)_0, (+,-,+)$	$\mathbb{Z}^{12}$	$\mathbb{Z}^3$	0	0
		$(-,-,-)_{1/2}, (-,+,-)$	$\mathbb{Z}^7$	$\mathbb{Z}$	$\mathbb{Z}^3$	0
		$(+,+,-), (+,-,-)$	$\mathbb{Z}^4$	$\mathbb{Z}^7$	0	0
		$(-,+,+), (-,-,+)$	$\mathbb{Z}$	$\mathbb{Z}^5 + \mathbb{Z}_2$	$\mathbb{Z}$	0
52	$Pnna$	all	$\mathbb{Z}^5$	$\mathbb{Z}^2$	0	0
53	$Pmna$	$(+,+,+)_0, (-,-,-)_{1/2}, (+,+,-), (-,-,+)$	$\mathbb{Z}^9$	$\mathbb{Z}$	$\mathbb{Z}$	0
		$(+,-,+), (-,+,+), (+,-,-), (-,+,-)$	$\mathbb{Z}^2$	$\mathbb{Z}^5$	0	0

(Cont.)

SG	Short	$(\epsilon(m_{100}, m_{010}), \epsilon(m_{100}, m_{001}), \epsilon(m_{010}, m_{001}))$	$E_2^{0,0}$	$E_2^{1,0}$	$E_2^{2,0}$	$E_2^{3,0}$
54	$Pcca$	$(+,+,+)_0, (+,+,-), (+,-,+), (+,-,-)$	$\mathbb{Z}^6$	$\mathbb{Z}^3$	0	0
		$(-,-,-)_{1/2}, (-,+,+), (-,+,-), (-,-,+)$	$\mathbb{Z}^4$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
55	$Pbam$	$(+,+,+)_0, (-,+,+)$	$\mathbb{Z}^9$	$\mathbb{Z}_2^3$	0	0
		$(-,-,-)_{1/2}, (+,-,-)$	$\mathbb{Z}^7$	0	$\mathbb{Z}^2+\mathbb{Z}_2$	0
		$(+,-,+), (+,+,-), (-,+,-), (-,-,+)$	$\mathbb{Z}$	$\mathbb{Z}^4+\mathbb{Z}_2$	0	0
56	$Pccn$	$(+,+,+)_0, (+,+,-), (+,-,+), (+,-,-)$	$\mathbb{Z}^5$	$\mathbb{Z}^2+\mathbb{Z}_2$	0	0
		$(-,-,-)_{1/2}, (-,+,+), (-,+,-), (-,-,+)$	$\mathbb{Z}^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
57	$Pbcm$	$(+,+,+)_0, (+,+,-), (-,+,+), (-,+,-)$	$\mathbb{Z}^5$	$\mathbb{Z}^2+\mathbb{Z}_2$	0	0
		$(-,-,-)_{1/2}, (+,-,+), (+,-,-), (-,-,+)$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}$	0
58	$Pnnm$	$(+,+,+)_0, (-,-,-)_{1/2}, (-,+,+), (+,-,-)$	$\mathbb{Z}^8$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
		$(+,+,-), (+,-,+), (-,+,-), (-,-,+)$	$\mathbb{Z}$	$\mathbb{Z}^4+\mathbb{Z}_2$	0	0
59	$Pmmn$	$(+,+,+)_0, (+,+,-), (+,-,+), (+,-,-)$	$\mathbb{Z}^7$	$\mathbb{Z}^4$	0	0
		$(-,-,-)_{1/2}, (-,+,+), (-,+,-), (-,-,+)$	$\mathbb{Z}^3$	$\mathbb{Z}^2+\mathbb{Z}_2$	$\mathbb{Z}^2$	0
60	$Pbcn$	all	$\mathbb{Z}^4$	$\mathbb{Z}+\mathbb{Z}_2$	0	0
61	$Pbca$	all	$\mathbb{Z}^3$	$\mathbb{Z}_2^2$	0	0
62	$Pnma$	$(+,+,+)_0, (+,-,+), (-,+,+), (-,-,+)$	$\mathbb{Z}^4$	$\mathbb{Z}+\mathbb{Z}_2^2$	0	0
		$(-,-,-)_{1/2}, (+,-,-), (-,+,-), (+,+,-)$	$\mathbb{Z}^3$	$\mathbb{Z}+\mathbb{Z}_2$	$\mathbb{Z}$	0
63	$Cmcm$	$(+,+,+)_0, (+,+,-)$	$\mathbb{Z}^8$	$\mathbb{Z}^2$	0	0
		$(-,-,-)_{1/2}, (-,-,+)$	$\mathbb{Z}^5$	$\mathbb{Z}$	$\mathbb{Z}^2$	0
		$(+,-,+), (+,-,-)$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}$	0
		$(-,+,+), (-,+,-)$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	0	0
64	$Cmca$	$(+,+,+)_0, (+,+,-)$	$\mathbb{Z}^7$	$\mathbb{Z}+\mathbb{Z}_2$	0	0
		$(-,-,-)_{1/2}, (-,-,+)$	$\mathbb{Z}^5$	0	$\mathbb{Z}+\mathbb{Z}_2$	0
		$(+,-,+), (+,-,-)$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	0
		$(-,+,+), (-,+,-)$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0
65	$Cmmm$	$(+,+,+)_0$	$\mathbb{Z}^{18}$	0	0	0
		$(-,-,-)_{1/2}$	$\mathbb{Z}^8$	0	$\mathbb{Z}^4$	0
		$(+,+,-), (+,-,+)$	$\mathbb{Z}^2$	$\mathbb{Z}^8$	0	0
		$(-,+,+)$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	0	0
		$(+,-,-)$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
		$(-,+,-), (-,-,+)$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}$	0
66	$Cccm$	$(+,+,+)_0, (+,-,-)$	$\mathbb{Z}^{11}$	0	$\mathbb{Z}$	0
		$(-,-,-)_{1/2}, (-,+,+)$	$\mathbb{Z}^7$	$\mathbb{Z}^2$	$\mathbb{Z}$	0
		$(+,+,-), (+,-,+)$	$\mathbb{Z}$	$\mathbb{Z}^7$	0	0
		$(-,+,-), (-,-,+)$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}_2$	0
67	$Cmma$	$(+,+,+)_0$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	0	0
		$(-,-,-)_{1/2}$	$\mathbb{Z}^5$	$\mathbb{Z}$	$\mathbb{Z}^2+\mathbb{Z}_2$	0
		$(+,+,-), (+,-,+)$	$\mathbb{Z}^5$	$\mathbb{Z}^5$	0	0
		$(-,+,+)$	$\mathbb{Z}$	$\mathbb{Z}^7+\mathbb{Z}_2$	0	0
		$(+,-,-)$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}_2^2$	0
		$(-,+,-), (-,-,+)$	$\mathbb{Z}^6$	$\mathbb{Z}$	$\mathbb{Z}$	0
68	$Ccca$	$(+,+,+)_0, (+,-,-)$	$\mathbb{Z}^7$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
		$(-,-,-)_{1/2}, (-,+,+)$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}_2$	0
		$(+,+,-), (+,-,+)$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	0	0
		$(-,+,-), (-,-,+)$	$\mathbb{Z}^6$	0	$\mathbb{Z}_2$	0
69	$Fmmm$	$(+,+,+)_0$	$\mathbb{Z}^{15}$	0	0	0
		$(-,-,-)_{1/2}$	$\mathbb{Z}^6$	0	$\mathbb{Z}^3+\mathbb{Z}_2$	0
		$(+,+,-), (+,-,+), (-,+,+)$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	0	0
		$(+,-,-), (-,+,-), (-,-,+)$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}$	0
70	$Fddd$	$(+,+,+)_0, (+,-,-), (-,+,-), (-,-,+)$	$\mathbb{Z}^6$	0	$\mathbb{Z}_2$	0
		$(-,-,-)_{1/2}, (+,+,-), (+,-,+), (-,+,+)$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0
71	$Immm$	$(+,+,+)_0$	$\mathbb{Z}^{15}$	0	0	0
		$(-,-,-)_{1/2}$	$\mathbb{Z}^6$	$\mathbb{Z}_2$	$\mathbb{Z}^3$	0
		$(+,+,-), (+,-,+), (-,+,+)$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	0	0
		$(+,-,-), (-,+,-), (-,-,+)$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}$	0

(Cont.)

another 5 pages …

Summary

- K-theory provides a systematic way to explore possible new topological phases.
- The band theory and space groups are naturally taken into account in the K-theory approach
- We have discovered many new topological numbers. So new topological phases should be discovered in near future.

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