Theoretical status of the muon $g - 2$

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Precision Physics, Fundamental Interactions and Structure of Matter

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Outline

- Basics of the anomalous magnetic moment
- Electron $g-2$, determination of α
- Muon $g-2$: QED, weak interactions, hadronic contributions
- Hadronic vacuum polarization (HVP)
- Hadronic light-by-light scattering (HLbL)
- A data-driven approach to HLbL using dispersion relations
- Precision of a data-driven approach to HLbL: pseudoscalar-pole contribution
- New Physics contributions to the muon $g-2$
- Conclusions and Outlook

Basics of the anomalous magnetic moment

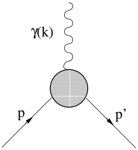
Electrostatic properties of charged particles:
Charge Q, Magnetic moment $\vec{\mu}$, Electric dipole moment $\vec{d}$

For a spin 1/2 particle:

$$\vec{\mu} = g\frac{e}{2m}\vec{s}, \qquad \underbrace{g = 2}_{\text{Dirac}}(1+a), \qquad a = \frac{1}{2}(g-2) : \text{anomalous magnetic moment}$$

Long interplay between experiment and theory: structure of fundamental forces

In Quantum Field Theory (with C,P invariance):



$$= \quad (-ie)\bar{u}(p')\left[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_\nu}{2m}\underbrace{F_2(k^2)}_{\text{Pauli}}\right]u(p)$$

$$F_1(0) = 1 \quad \text{and} \quad F_2(0) = a$$

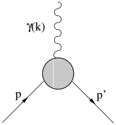
a_e: Test of QED. Most precise determination of $\alpha = e^2/4\pi$.
a_μ: Less precisely measured than a_e, but all sectors of Standard Model (SM), i.e. QED, Weak and QCD (hadronic), contribute significantly.
Sensitive to possible contributions from New Physics. Often (but not always !):

$$a_\ell \sim \left(\frac{m_\ell}{m_{\text{NP}}}\right)^2 \Rightarrow \left(\frac{m_\mu}{m_e}\right)^2 \sim 43000 \text{ more sensitive than } a_e \text{ [exp. precision} \to \text{factor 19]}$$

Anomalous magnetic moment in quantum field theory

Quantized spin 1/2 particle interacting with external, classical electromagnetic field

4 form factors in vertex function
(momentum transfer $k = p' - p$, not assuming parity or charge conjugation invariance)



$$\equiv \quad i\langle p', s'|j^\mu(0)|p, s\rangle$$

$$= \quad (-ie)\bar{u}(p', s')\left[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_\nu}{2m}\underbrace{F_2(k^2)}_{\text{Pauli}} + \gamma^5\frac{\sigma^{\mu\nu}k_\nu}{2m}F_3(k^2) + \gamma^5\left(k^2\gamma^\mu - \not{k}\, k^\mu\right)F_4(k^2)\right]u(p, s)$$

$\not{k} = \gamma^\mu k_\mu$. Real form factors for spacelike $k^2 \leq 0$. Non-relativistic, static limit:

$$F_1(0) = 1 \quad \text{(renormalization of charge } e\text{)}$$

$$\mu = \frac{e}{2m}(F_1(0) + F_2(0)) \quad \text{(magnetic moment)}$$

$$a = F_2(0) \quad \text{(anomalous magnetic moment)}$$

$$d = -\frac{e}{2m}F_3(0) \quad \text{(electric dipole moment, violates P and CP)}$$

$$F_4(0) = \text{anapole moment (violates P)}$$

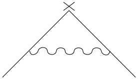
Some theoretical comments

- Anomalous magnetic moment is finite and calculable
  Corresponds to effective interaction Lagrangian of mass dimension 5:

$$\mathcal{L}_{\text{eff}}^{\text{AMM}} = -\frac{e_\ell a_\ell}{4m_\ell} \bar{\psi}(x)\sigma^{\mu\nu}\psi(x)F_{\mu\nu}(x)$$

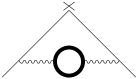
  $a_\ell = F_2(0)$ can be calculated unambiguously in renormalizable QFT, since there is no counterterm to absorb potential ultraviolet divergence.

- Anomalous magnetic moments are dimensionless
  To lowest order in perturbation theory in quantum electrodynamics (QED):



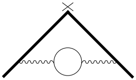
$$= a_e = a_\mu = \frac{\alpha}{2\pi}$$ [Schwinger '47/'48]

- Loops with different masses $\Rightarrow a_e \neq a_\mu$
  - Internal large masses decouple (not always !):



$$= \left[\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 + \mathcal{O}\left(\frac{m_e^4}{m_\mu^4}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  - Internal small masses give rise to large log's of mass ratios:



$$= \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

Electron $g-2$

Electron $g-2$: Theory

Main contribution in Standard Model (SM) from mass-independent Feynman diagrams in QED with electrons in internal lines (perturbative series in α):

$$
\begin{aligned}
a_e^{\text{SM}} \quad = \quad & \sum_{n=1}^{5} c_n \left(\frac{\alpha}{\pi}\right)^n \\
& +2.7478(2) \times 10^{-12} \quad \text{[Loops in QED with } \mu, \tau\text{]} \\
& +0.0297(5) \times 10^{-12} \quad \text{[weak interactions]} \\
& +1.706(15) \times 10^{-12} \quad \text{[strong interactions / hadrons]}
\end{aligned}
$$

The numbers are from Aoyama et al. '15.

QED: mass-independent contributions to a_e

- α: 1-loop, 1 Feynman diagram; Schwinger '47/'48:

  $c_1 = \frac{1}{2}$

- α^2: 2-loops, 7 Feynman diagrams; Petermann '57, Sommerfield '57:

  $c_2 = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3) = -0.32847896557919378\ldots$

- α^3: 3-loops, 72 Feynman diagrams; ..., Laporta, Remiddi '96:

$$\begin{aligned} c_3 &= \frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2\ln 2 + \frac{139}{18}\zeta(3) - \frac{239}{2160}\pi^4 \\ &\quad + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2\right\} \\ &= 1.181241456587\ldots \end{aligned}$$

- α^4: 4-loops, 891 Feynman diagrams; Kinoshita et al. '99, ..., Aoyama et al. '08; '12, '15:

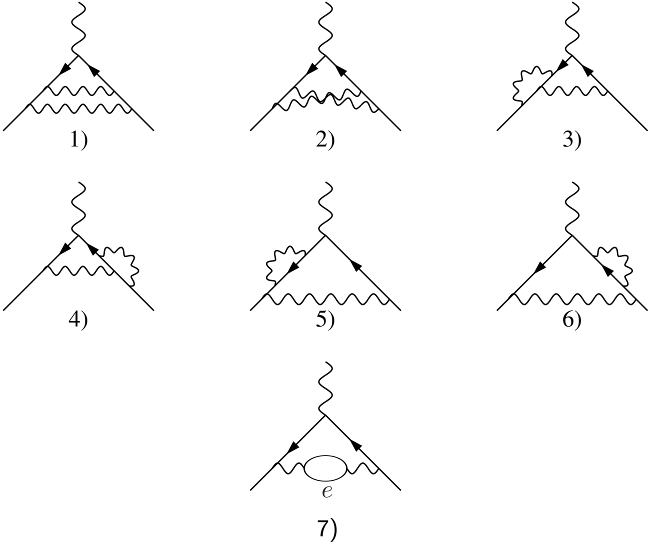
  $c_4 = -1.91298(84)$ (numerical evaluation)

- α^5: 5-loops, 12672 Feynman diagrams; Aoyama et al. '05, ..., '12, '15:

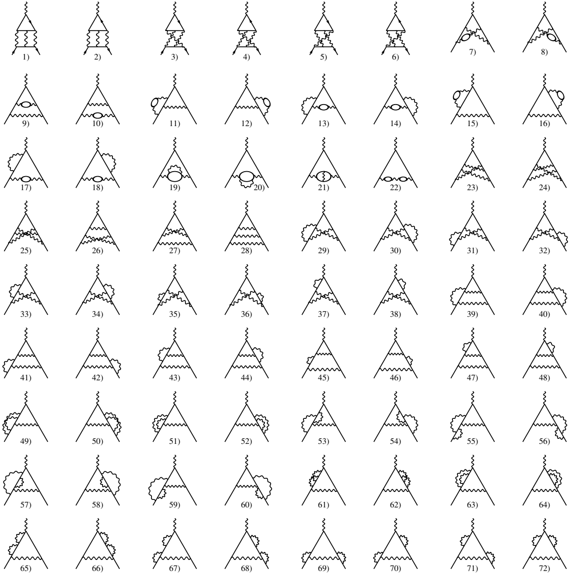
  $c_5 = 7.795(336)$ (numerical evaluation)

  Replaces earlier rough estimate $c_5 = 0.0 \pm 4.6$.
  Result removes biggest theoretical uncertainty in a_e !

Mass-independent 2-loop Feynman diagrams in a_e



Mass-independent 3-loop Feynman diagrams in a_e



Determination of fine-structure constant α from $g-2$ of electron

- Recent measurement of α via recoil-velocity of Rubidium atoms in atom interferometer (Bouchendira et al. '11 and recent CODATA input):
$$\alpha^{-1}(\text{Rb}) = 137.035\,999\,049(90) \quad [0.66\text{ppb}]$$
This leads to (Aoyama et al. '15):
$$a_e^{\text{SM}}(\text{Rb}) = 1\,159\,652\,181.643\,\underbrace{(25)}_{c_4}\,\underbrace{(23)}_{c_5}\,\underbrace{(16)}_{\text{had}}\,\underbrace{(763)}_{\alpha(\text{Rb})}\,[764] \times 10^{-12} \quad [0.67\text{ppb}]$$
$$\Rightarrow a_e^{\text{exp}} - a_e^{\text{SM}}(\text{Rb}) = -0.91(0.82) \times 10^{-12} \quad [\text{Error from } \alpha(\text{Rb}) \text{ dominates !}]$$
$\rightarrow$ Test of QED !

- Use a_e^{exp} to determine α from series expansion in QED (contributions from weak and strong interactions under control !). Assume: Standard Model "correct", no New Physics (Aoyama et al. '15):
$$\alpha^{-1}(a_e) = 137.035\,999\,1570\,\underbrace{(29)}_{c_4}\,\underbrace{(27)}_{c_5}\,\underbrace{(18)}_{\text{had+EW}}\,\underbrace{(331)}_{a_e^{\text{exp}}}\,[334] \quad [0.25\text{ppb}]$$
The uncertainty from theory has been improved considerably by Aoyama et al. '12, '15, the experimental uncertainty in a_e^{exp} is now the limiting factor.

- Today the most precise determination of the fine-structure constant α, a fundamental parameter of the Standard Model.

Muon $g - 2$

Milestones in measurements of a_μ

Authors	Lab	Muon Anomaly	
Garwin et al. '60	CERN	0.001 13(14)	
Charpak et al. '61	CERN	0.001 145(22)	
Charpak et al. '62	CERN	0.001 162(5)	
Farley et al. '66	CERN	0.001 165(3)	
Bailey et al. '68	CERN	0.001 166 16(31)	
Bailey et al. '79	CERN	0.001 165 923 0(84)	
Brown et al. '00	BNL	0.001 165 919 1(59)	(μ^+)
Brown et al. '01	BNL	0.001 165 920 2(14)(6)	(μ^+)
Bennett et al. '02	BNL	0.001 165 920 4(7)(5)	(μ^+)
Bennett et al. '04	BNL	0.001 165 921 4(8)(3)	(μ^-)

World average experimental value (dominated by $g-2$ Collaboration at BNL, Bennett et al. '06 + CODATA 2008 value for $\lambda = \mu_\mu/\mu_p$):

$$a_\mu^{\mathrm{exp}} = (116\,592\,089 \pm 63) \times 10^{-11} \quad [0.5\mathrm{ppm}]$$

Goal of new planned $g-2$ experiments: $\delta a_\mu = 16 \times 10^{-11}$

Fermilab E989: partly recycled from BNL: moved ring magnet ! (http://muon-g-2.fnal.gov/bigmove/) First beam in 2017, should reach this precision by 2020. J-PARC E34: completely new concept with low-energy muons, not magic γ. Aims in Phase 1 for about $\delta a_\mu = 45 \times 10^{-11}$.

Theory needs to match this precision !

For comparison: Electron (stable !) (Hanneke et al. '08):

$$a_e^{\mathrm{exp}} = (1\,159\,652\,180.73 \pm 0.28) \times 10^{-12} \quad [0.24\mathrm{ppb}]$$

Muon $g-2$: Theory

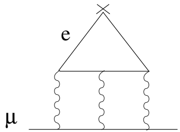
In Standard Model (SM):

$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{had}}$$

In contrast to a_e, here now the contributions from weak and strong interactions (hadrons) are relevant, since $a_\mu \sim (m_\mu/M)^2$.

QED contributions

- Diagrams with internal electron loops are enhanced.
- At 2-loops: vacuum polarization from electron loops enhanced by QED short-distance logarithm
- At 3-loops: light-by-light scattering from electron loops enhanced by QED infrared logarithm [Aldins et al. '69, '70; Laporta, Remiddi '93]



$$+ \ldots \quad a_\mu^{(3)}\Big|_{\text{lbyl}} = \left[\frac{2}{3}\pi^2 \ln \frac{m_\mu}{m_e} + \ldots\right] \left(\frac{\alpha}{\pi}\right)^3 = 20.947\ldots \left(\frac{\alpha}{\pi}\right)^3$$

- Loops with tau's suppressed (decoupling)

QED result up to 5 loops

Include contributions from all leptons (Aoyama et al. '12):

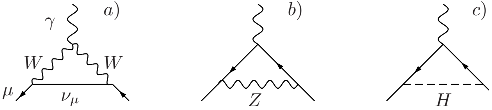
$$
\begin{aligned}
a_\mu^{\text{QED}} &= 0.5 \times \left(\frac{\alpha}{\pi}\right) + 0.765\,857\,425 \underbrace{(17)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^2 \\
&\quad + 24.050\,509\,96 \underbrace{(32)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^3 + 130.8796 \underbrace{(63)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^4 \\
&\quad + 753.29 \underbrace{(1.04)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^5 \\
&= 116\,584\,718.853 \underbrace{(9)}_{m_\mu/m_{e,\tau}} \underbrace{(19)}_{c_4} \underbrace{(7)}_{c_5} \underbrace{(29)}_{\alpha(a_e)} [36] \times 10^{-11}
\end{aligned}
$$

- 4-loop: analytical results for electron and tau-loops (asymptotic expansions) by Steinhauser et al. '15 + '16.
- Earlier evaluation of 5-loop contribution yielded $c_5 = 662(20)$ (Kinoshita, Nio '06, numerical evaluation of 2958 diagrams, known or likely to be enhanced). New value is 4.5σ from this leading log estimate and 20 times more precise.
- Aoyama et al. '12: What about the 6-loop term ? Leading contribution from light-by-light scattering with electron loop and insertions of vacuum-polarization loops of electrons into each photon line $\Rightarrow a_\mu^{\text{QED}}(\text{6-loops}) \sim 0.1 \times 10^{-11}$

Contributions from weak interaction

Numbers from recent reanalysis by Gnendiger et al. '13.

1-loop contributions [Jackiw + Weinberg, 1972; ...]:



$$a_\mu^{\text{weak, (1)}}(W) = \frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\frac{10}{3} + \mathcal{O}(m_\mu^2/M_W^2) = 388.70 \times 10^{-11}$$

$$a_\mu^{\text{weak, (1)}}(Z) = \frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\frac{(-1+4s_W^2)^2 - 5}{3} + \mathcal{O}(m_\mu^2/M_Z^2) = -193.89 \times 10^{-11}$$

Contribution from Higgs negligible: $a_\mu^{\text{weak, (1)}}(H) \leq 5 \times 10^{-14}$ for $m_H = 126$ GeV.

$$a_\mu^{\text{weak, (1)}} = (194.80 \pm 0.01) \times 10^{-11}$$

2-loop contributions (1678 diagrams) [Czarnecki et al. '95, '96; ...]:

$$a_\mu^{\text{weak, (2)}} = (-41.2 \pm 1.0) \times 10^{-11}, \quad \text{large since} \sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$$

Total weak contribution:

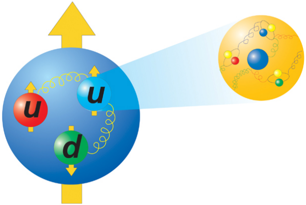
$$a_\mu^{\text{weak}} = (153.6 \pm 1.0) \times 10^{-11}$$

Under control ! With knowledge of $M_H = 125.6 \pm 1.5$ GeV, uncertainty now mostly hadronic $\pm 1.0 \times 10^{-11}$ (Peris et al. '95; Knecht et al. '02; Czarnecki et al. '03, '06).
3-loop effects via RG: $\pm 0.20 \times 10^{-11}$ (Degrassi, Giudice '98; Czarnecki et al. '03).

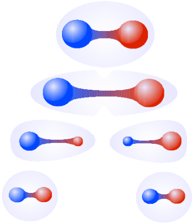
Hadronic contributions to $g-2$

The strong interactions (Quantum Chromodynamics)

- Strong interactions: quantum chromodynamics (QCD) with quarks and gluons
- Observed particles in Nature: Hadrons
  1. Mesons (quark + antiquark: $q\bar{q}$): $\pi, K, \eta, \rho, \ldots$
  2. Baryons (3 quarks: qqq): $p, n, \Lambda, \Sigma, \Delta, \ldots$
- Cannot describe hadrons in series expansion in strong coupling constant of QCD with $\alpha_s(E = m_{\text{proton}}) \approx 0.5$.
  Particularly true for light hadrons which consist of three lightest quarks u, d, s.
  Non-perturbative effects like "confinement" of quarks and gluons inside hadrons.



Source: NIKHEF



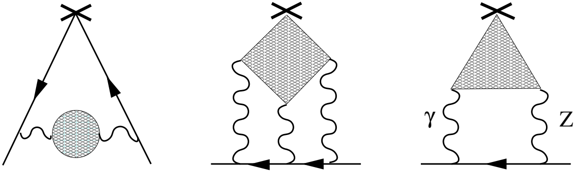
Source: NIC Jülich

- Possible approaches to QCD at low energies:
  1. Lattice QCD: limited applications, often still limited precision
  2. Effective quantum field theories with hadrons (chiral perturbation theory): limited validity
  3. Simplifying hadronic models: model uncertainties not controllable
  4. Dispersion relations: extend validity of EFT's, reduce model dependence, often not all the needed input data available

Hadronic contributions to the muon $g-2$

Largest source of uncertainty in theoretical prediction of a_μ !

Different types of contributions:



(a) (b) (c)

Light quark loop not well defined → Hadronic "blob"

(a) Hadronic vacuum polarization $\mathcal{O}(\alpha^2), \mathcal{O}(\alpha^3), \mathcal{O}(\alpha^4)$

(b) Hadronic light-by-light scattering $\mathcal{O}(\alpha^3), \mathcal{O}(\alpha^4)$

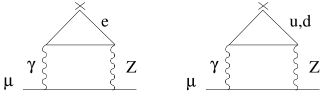
(c) 2-loop electroweak contributions $\mathcal{O}(\alpha G_F m_\mu^2)$

2-Loop EW

Small hadronic uncertainty from triangle diagrams.

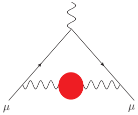
Anomaly cancellation within each generation !

Cannot separate leptons and quarks !

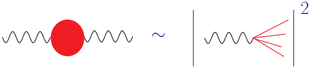


Hadronic vacuum polarization (HVP)

Hadronic vacuum polarization

$a_\mu^{\text{HVP}} =$ 

Optical theorem (from unitarity; conservation of probability) for hadronic contribution $\rightarrow$ dispersion relation:

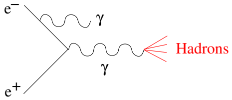
Im 
$\sim \quad \sigma(e^+e^- \to \gamma^* \to \text{hadrons})$

$$a_\mu^{\text{HVP}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s} K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

[Bouchiat, Michel '61; Durand '62; Brodsky, de Rafael '68; Gourdin, de Rafael '69]

$K(s)$ slowly varying, positive function $\Rightarrow$ a_μ^{HVP} positive. Data for hadronic cross section σ at low center-of-mass energies $\sqrt{s}$ important due to factor $1/s$: $\sim 70\%$ from $\pi\pi$ [$\rho(770)$] channel, $\sim 90\%$ from energy region below 1.8 GeV.

Other method instead of energy scan: "Radiative return" at colliders with fixed center-of-mass energy (DAΦNE, B-Factories, BEPC) [Binner et al. '99; Czyż et al. '00-'03]

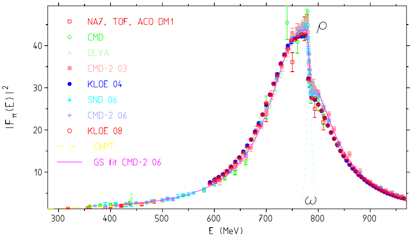


Measured hadronic cross-section

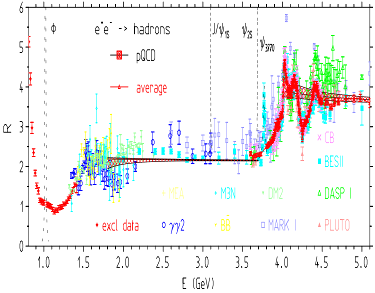
Pion form factor $|F_\pi(E)|^2$ ($\pi\pi$-channel)

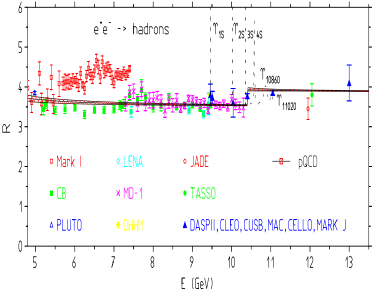
$$R(s) = \frac{1}{4}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}}|F_\pi(s)|^2$$

$$(4m_\pi^2 < s < 9m_\pi^2)$$



R-ratio:





Jegerlehner, AN '09

Hadronic vacuum polarization: some recent evaluations

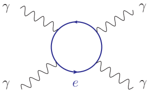
Authors	Contribution to $a_\mu^{\rm HVP} \times 10^{11}$
Jegerlehner '08; Jegerlehner, AN '09 (e^+e^-)	6903.0 ± 52.6
Davier et al. '09 (e^+e^-) [$+\ \tau$]	6955 ± 41 [7053 ± 45]
Teubner et al. '09 (e^+e^-)	6894 ± 40
Davier et al. '11, '14 (e^+e^-) [$+\ \tau$]	6923 ± 42 [7030 ± 44]
Jegerlehner, Szafron '11 (e^+e^-) [$+\ \tau$]	6907.5 ± 47.2 [6909.6 ± 46.5]
Hagiwara et al. '11 (e^+e^-)	6949.1 ± 42.7
Benayoun at al. '15 ($e^+e^- + \tau$: BHLS improved)	6818.6 ± 32.0
Jegerlehner '15 (e^+e^-) [$+\ \tau$]	6885.7 ± 42.8 [6889.1 ± 35.2]

- Precision: $< 1\%$. Non-trivial because of radiative corrections (radiated photons).
- Even if values for $a_\mu^{\rm HVP}$ after integration agree quite well, the systematic differences of a few % in the shape of the spectral functions from different experiments (BABAR, BES III, CMD-2, KLOE, SND) indicate that we do not yet have a complete understanding.
- Use of τ data: additional sources of isospin violation ? Ghozzi, Jegerlehner '04; Benayoun et al. '08, '09; Wolfe, Maltman '09; Jegerlehner, Szafron '11 ($\rho - \gamma$-mixing), also included in Jegerlehner '15 and in BHLS-approach by Benayoun et al. '15 (additional BHLS model uncertainty can lead to maximal shift in central value of $^{+15}_{-27} \times 10^{-11}$).
- Lattice QCD: Various groups are working on it, precision at level of about 3-5% (systematics dominated), not yet competitive with phenomenological evaluations.

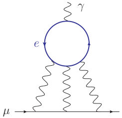
Hadronic light-by-light scattering (HLbL)

Hadronic light-by-light scattering in the muon $g-2$

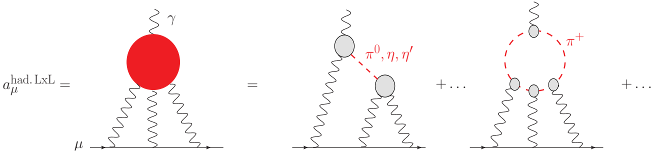
QED: light-by-light scattering at higher orders in perturbation series via lepton-loop:



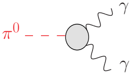
In muon $g-2$:
$\Rightarrow$



Hadronic light-by-light scattering in muon $g-2$ from strong interactions (QCD):



Coupling of photons to hadrons, e.g. π^0, via form factor:



View before 2014: in contrast to HVP, no direct relation to experimental data $\rightarrow$ size and even sign of contribution to a_μ unknown !
Approach: use hadronic model at low energies with exchanges and loops of resonances and some (dressed) "quark-loop" at high energies.
Problems: Four-point function depends on several invariant momenta $\Rightarrow$ distinction between low and high energies not as easy as for two-point function in HVP.
Mixed regions: one loop momentum Q_1^2 large, the other Q_2^2 small and vice versa.

HLbL in muon $g - 2$

- Only model calculations for total HLbL contribution: large uncertainties, difficult to control.
- Frequently used estimates:

$$a_\mu^{\mathrm{HLbL}} = (105 \pm 26) \times 10^{-11} \quad \text{(Prades, de Rafael, Vainshtein '09)}$$
$$a_\mu^{\mathrm{HLbL}} = (116 \pm 40) \times 10^{-11} \quad \text{(AN '09; Jegerlehner, AN '09)}$$

  Based almost on same input: calculations by various groups using different models for individual contributions. Error estimates are mostly guesses !
- Need much better understanding of complicated hadronic dynamics to get reliable error estimate of $\pm 20 \times 10^{-11}$ (δa_μ(future exp) $= 16 \times 10^{-11}$).
- Recent new proposal: Colangelo et al. '14, '15; Pauk, Vanderhaeghen '14: use dispersion relations (DR) to connect contribution to HLbL from presumably numerically dominant light pseudoscalars to in principle measurable form factors and cross-sections:

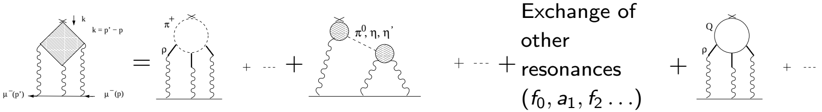
$$\gamma^*\gamma^* \rightarrow \pi^0, \eta, \eta'$$
$$\gamma^*\gamma^* \rightarrow \pi^+\pi^-, \pi^0\pi^0$$

  Could connect HLbL uncertainty to exp. measurement errors, like HVP.
  Note: no data yet with two off-shell photons !
- Future: HLbL from Lattice QCD.
  First steps and results: Blum et al. (RBC-UKQCD) '05, ..., '15, '16.
  Work ongoing by Mainz group: Green et al. '15; Asmussen et al. '16.

General approach to HLbL in muon $g-2$

Classification of de Rafael '94

Chiral counting p^2 (from Chiral Perturbation Theory (ChPT)) and large-N_C counting as guideline to classify contributions (all higher orders in p^2 and N_C contribute):



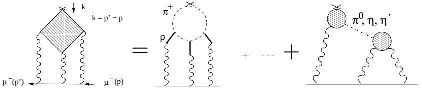
Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C
	pion-loop (dressed)	pseudoscalar exchanges		quark-loop (dressed)

Relevant scales in HLbL ($\langle VVVV \rangle$ with off-shell photons !): $0-2$ GeV $\gg m_\mu$!

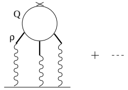
Constrain models using experimental data (processes of hadrons with photons: decays, form factors, scattering) and theory (ChPT at low energies; short-distance constraints from pQCD / OPE at high momenta).

General analysis of four-point function $\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3)$ relevant for $g-2$: Bijnens et al. '96; Bijnens, Talk at $g-2$ Workshop, Mainz, '14; Bijnens + Relefors '15 + 16; Eichmann et al. '14, '15; Colangelo et al. '15.

HLbL scattering: summary of selected results



+ ··· + Exchange of other resonances ($f_0, a_1, f_2 \ldots$) +



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

BPP: +83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS: +90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN: +80 (40)		+83 (12)		
MV: +136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007: +110 (40)				
PdRV: +105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN: +116 (40)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
	ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09 (compilation; "Glasgow consensus"); N,JN = AN '09; Jegerlehner, AN '09 (compilation)

Pseudoscalars: numerically dominant contribution (according to most models !).

Recall (in units of 10^{-11}): $\delta a_\mu(\mathrm{HVP}) \approx 45$; $\delta a_\mu(\exp\ [\mathrm{BNL}]) = 63$; $\delta a_\mu(\text{future exp}) = 16$

HLbL: recent developments

- New estimates for axial vectors (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15):
$$a_\mu^{\mathrm{HLbL;axial}} = (8 \pm 3) \times 10^{-11}$$
Substantially smaller than in MV '04 !
Would shift central values of compilations downwards:
$$a_\mu^{\mathrm{HLbL}} = (98 \pm 26) \times 10^{-11} \quad \text{(PdRV '09)}$$
$$a_\mu^{\mathrm{HLbL}} = (102 \pm 40) \times 10^{-11} \quad \text{(N, JN '09)}$$
- First estimate for tensor mesons (Pauk, Vanderhaeghen '14):
$$a_\mu^{\mathrm{HLbL;tensor}} = 1 \times 10^{-11}$$
- Open problem: Dressed pion-loop
Potentially important effect from pion polarizability and a_1 resonance
(Engel, Patel, Ramsey-Musolf '12; Engel '13; Engel, Ramsey-Musolf '13):
$$a_\mu^{\mathrm{HLbL;\pi-loop}} = -(11 - 71) \times 10^{-11}$$
Maybe large negative contribution, in contrast to BPP '96, HKS '96.
Not confirmed by recent reanalysis by Bijnens + Relefors '15, '16. Essentially get again old central value from BPP, but smaller error estimate:
$$a_\mu^{\mathrm{HLbL;\pi-loop}} = (-20 \pm 5) \times 10^{-11}$$
- Open problem: Dressed quark-loop
Dyson-Schwinger equation approach (Fischer, Goecke, Williams '11, '13):
$$a_\mu^{\mathrm{HLbL;quark-loop}} = 107 \times 10^{-11} \qquad \text{(still incomplete !)}$$
Large contribution, no damping seen, in contrast to BPP '96, HKS '96.

A data-driven approach to HLbL using dispersion relations

Data-driven approach to HLbL using dispersion relations (DR)

Strategy: Split contributions to HLbL into two parts:

I: Data-driven evaluation using DR (hopefully numerically dominant):

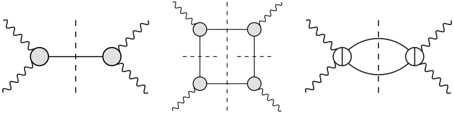
(1) π^0, η, η' poles
(2) $\pi\pi$ intermediate state

II: Model dependent evaluation (hopefully numerically subdominant):

(1) Axial vectors (3π-intermediate state), ...
(2) Quark-loop, matching with pQCD

Error goals: Part I: 10% precision (data driven), Part II: 30% precision.
To achieve overall error of about 20% ($\delta a_\mu^{\text{HLbL}} = 20 \times 10^{-11}$).

Colangelo et al. '14, '15:
Classify intermediate states in 4-point function. Then project onto $g-2$.

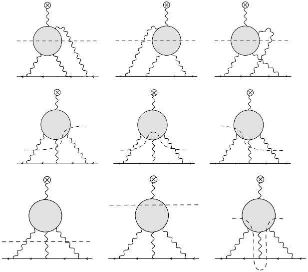


Evaluation of pion-box contribution (middle diagram) using precise information on pion vector form factor:

$$a_\mu^{\text{FsQED}} = -15.9 \times 10^{-11}$$

Colangelo, talk at Radio Monte Carlo Meeting, Frascati, May 2016. Error analysis ongoing, around $\pm 0.5 \times 10^{-11}$.

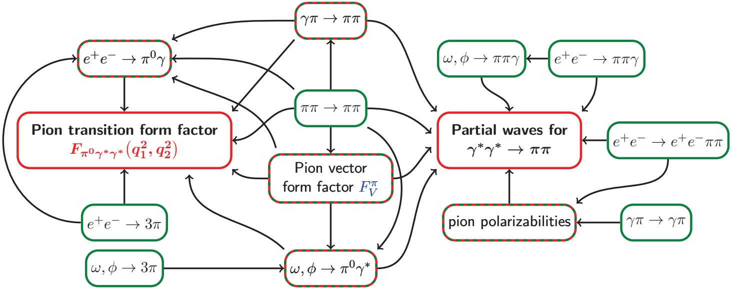
Pauk, Vanderhaeghen '14:
Write DR directly for Pauli form factor $F_2(k^2)$.



Data-driven approach to HLbL using dispersion relations (DR) (continued)

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer arXiv:1408.2517 (PLB '14)



Artwork by M. Hoferichter

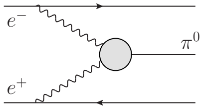
A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists

From talk by Colangelo at Radio Monte Carlo Meeting, Frascati, May 2016

Experimental data on hadronic $\gamma\gamma \to \gamma\gamma$

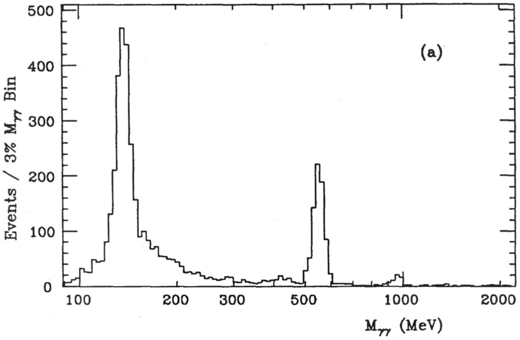
In any case, it is a good idea to look at actual data. For instance, obtained by Crystal Ball detector '88 via the process:

$$e^+e^- \to e^+e^-\gamma^*\gamma^* \to e^+e^-\pi^0$$



Feynman diagram from Colangelo et al., arXiv:1408.2517

Invariant $\gamma\gamma$ mass spectrum:



Three spikes from light pseudoscalars in reaction:

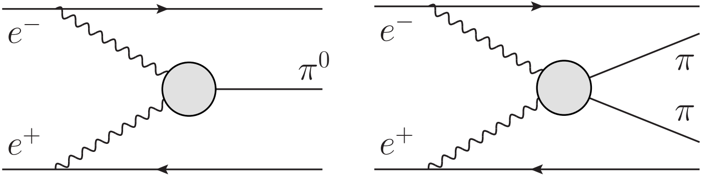
$$\gamma\gamma \to \pi^0, \eta, \eta' \to \gamma\gamma$$

for (almost) real photons.

Photon-photon processes in e^+e^- collisions

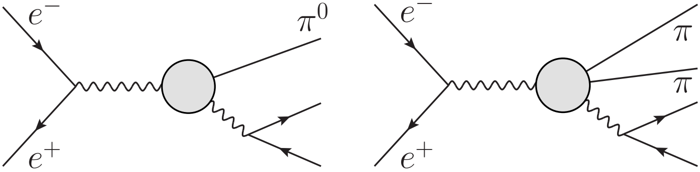
Feynman diagrams from Colangelo et al., arXiv:1408.2517

Space-like kinematics:



By tagging the outgoing leptons (single-tag, double-tag), one can infer the virtual (space-like) momenta $Q_i^2 = -q_i^2$ of the photons.
Left: process allows to measure pion-photon-photon transition form factor (TFF) $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$.

Time-like kinematics:



Precision of a data-driven approach to HLbL: pseudoscalar-pole contribution

$a_\mu^{\mathrm{HLbL;P}}, \mathrm{P} = \pi^0, \eta, \eta'$: impact of precision of form factor measurements

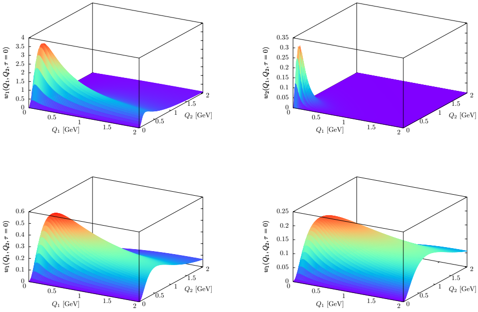
AN '16

In Jegerlehner, AN '09, a 3-dimensional integral representation for the pseudoscalar-pole contribution was derived. Schematically:

$$a_\mu^{\mathrm{HLbL;P}} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sum_i w_i(Q_1, Q_2, \tau)\, f_i(Q_1, Q_2, \tau)$$

with universal weight functions w_i (for Euclidean (space-like) momenta: $Q_1 \cdot Q_2 = |Q_1||Q_2|\tau, \tau = \cos\theta$). Dependence on form factors resides in the f_i.

Weight functions w_i:



Top: weight functions $w_{1,2}(Q_1, Q_2, \tau)$ for π^0 with $\theta = 90^\circ (\tau = 0)$.

Bottom: weight functions $w_1(Q_1, Q_2, \tau)$ for η (left) and η' (right).

- Relevant momentum regions below 1 GeV for π^0, below 1.5 GeV for η, η'.
- Analysis of current and future measurement precision of single-virtual $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q^2, 0)$ and double-virtual transition form factor $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$, based on Monte Carlo study for BES III by Denig, Redmer, Wasser.
- Data-driven precision for HLbL pseudoscalar-pole contribution that could be achieved in a few years:

$$\delta a_\mu^{\mathrm{HLbL};\pi^0} / a_\mu^{\mathrm{HLbL};\pi^0} = 14\%$$
$$\delta a_\mu^{\mathrm{HLbL};\eta} / a_\mu^{\mathrm{HLbL};\eta} = 23\%$$
$$\delta a_\mu^{\mathrm{HLbL};\eta'} / a_\mu^{\mathrm{HLbL};\eta'} = 15\%$$

Pseudoscalar contribution to HLbL

- Most calculations for neutral pion and all light pseudoscalars π^0, η, η' agree at level of 15%, but full range of estimates (central values) much larger:

$$a_\mu^{\mathrm{HLbL};\pi^0} = (50 - 80) \times 10^{-11} = (65 \pm 15) \times 10^{-11} \ (\pm 23\%)$$

$$a_\mu^{\mathrm{HLbL;P}} = (59 - 114) \times 10^{-11} = (87 \pm 27) \times 10^{-11} \ (\pm 31\%)$$

- Study precision which could be reached with data-driven estimate of pseudoscalar-pole contribution to HLbL (AN '16)

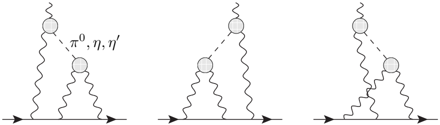
- Relevant momentum regions where data on doubly off-shell transition form factor $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ will be needed from direct experimental measurements, via DR for form factor itself (Hoferichter et al. '14) or from Lattice QCD (Gérardin, Meyer, AN '16), to better control this numerically dominant contribution to HLbL and its uncertainty.

- Impact on precision of $a_\mu^{\mathrm{HLbL;P}}$ based on estimated experimental uncertainties of $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ obtained from process

$$e^+e^- \to e^+e^-\gamma^*\gamma^* \to e^+e^-\mathrm{P}$$

using results from Monte Carlo simulation for BESIII (Mainz group: Denig, Redmer, Wasser).

Pion-pole contribution (analogously for η, η')



3-dimensional integral representation (Jegerlehner, AN '09):

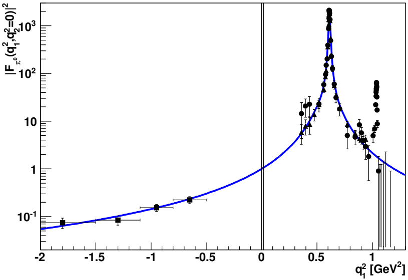
$$a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right]$$

$$a_\mu^{\mathrm{HLbL};\pi^0(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau\, w_1(Q_1, Q_2, \tau) \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2) \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)$$

$$a_\mu^{\mathrm{HLbL};\pi^0(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau\, w_2(Q_1, Q_2, \tau) \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

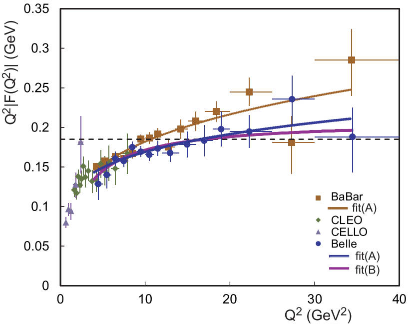
- After Wick rotation: Q_1, Q_2 are Euclidean (spacelike) four-momenta. Integrals run over the lengths of the four-vectors with $Q_i \equiv |(Q_i)_\mu|, i = 1, 2$ and angle θ between them: $Q_1 \cdot Q_2 = Q_1 Q_2 \cos\theta$, $\tau = \cos\theta$.
- Separation of generic kinematics described by model-independent weight functions $w_{1,2}(Q_1, Q_2, \tau)$ and double-virtual form factors $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ which can in principle be measured.
- Only single-virtual form factor $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(q^2, 0)$ in spacelike and timelike region has been measured so far. No data yet for spacelike $Q^2 \leq 0.5$ GeV^2.
- Measurement of double-virtual form factor $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ in spacelike region planned at BES III down to $Q^2 = 0.3$ GeV^2.

Experimental data for the π^0 transition form factor



$$|\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2 = 0)|^2$$

Single-virtual π^0 transition form factor in the low $|q^2|$ region from SND '00 and CMD-2 '04 data (timelike, $q_1^2 > 0$) from the reaction $e^+e^- \to \pi^0\gamma$ and CELLO '91 data (spacelike, $q_1^2 < 0$) from $e^+e^- \to e^+e^-\gamma^*\gamma^* \to e^+e^-\pi^0$ (plot from Czerwinski et al., 1207.6556).

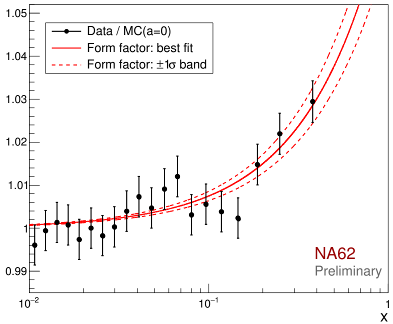


$$Q^2F(Q^2) \equiv Q^2\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, 0)$$

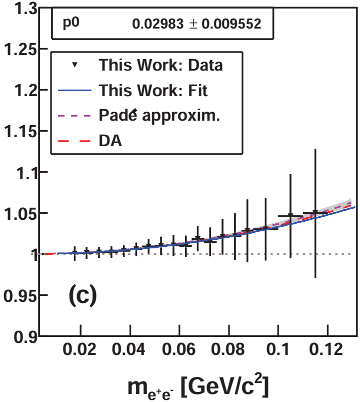
Collection of data for the single-virtual π^0 transition form factor in the spacelike region from CELLO '91, CLEO '98, BaBar '09, Belle '12 (plot from Belle '12).

Experimental data for the π^0 transition form factor (continued)

Very recently: first reliable measurements of TFF in timelike region in single Dalitz decays $\pi^0 \to e^+ e^- \gamma$

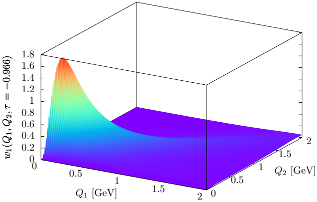


TFF/Data as function of $x = (M_{ee}/m_{\pi^0})^2$
(from Goudzovski (for NA62 Collaboration), 1609.02952)

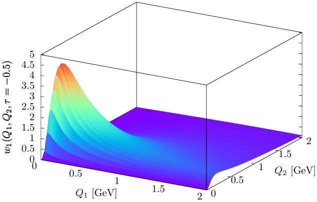


$|\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q^2, 0)|^2$ as function of the invariant mass, $|q| = m_{e^+e^-}$
(A2 Collaboration, 1611.04739)

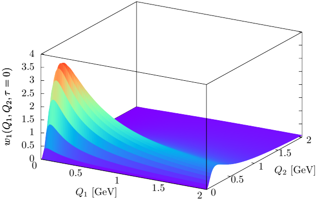
Weight function $w_1(Q_1, Q_2, \tau)$ for π^0



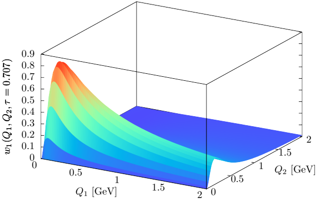
$\tau = -0.966, \quad \theta = 165^\circ$



$\tau = -0.5, \quad \theta = 120^\circ$



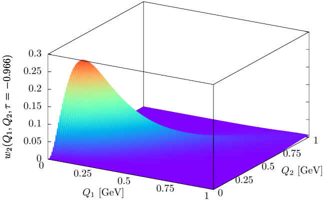
$\tau = 0, \quad \theta = 90^\circ$



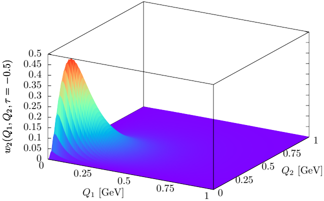
$\tau = 0.707, \quad \theta = 45^\circ$

Low momentum region most important. Peak around $Q_1 \sim 0.2$ GeV, $Q_2 \sim 0.15$ GeV.
Slopes along the two axis and along the diagonal (at $Q_1 = Q_2 = 0$) vanish.
For $\tau > -0.85$ ($\theta < 150^\circ$) a ridge develops along Q_1 direction for $Q_2 \sim 0.2$ GeV.
Leads for constant form factor to a divergence $\ln^2 \Lambda$ for some momentum cutoff Λ.
Realistic form factor falls off for large Q_i and integral $a_\mu^{\mathrm{HLbL};\pi^0(1)}$ will be convergent.

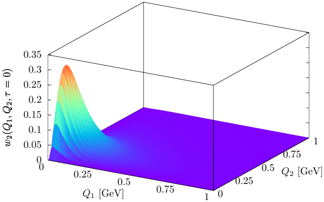
Weight function $w_2(Q_1, Q_2, \tau)$ for π^0



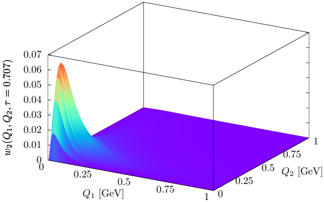
$\tau = -0.966, \quad \theta = 165^\circ$



$\tau = -0.5, \quad \theta = 120^\circ$



$\tau = 0, \quad \theta = 90^\circ$



$\tau = 0.707, \quad \theta = 45^\circ$

w_2 about a factor 10 smaller than w_1.
No ridge in one direction, since $w_2(Q_1, Q_2, \tau)$ is symmetric under $Q_1 \leftrightarrow Q_2$.
Peak for $Q_1 = Q_2 \sim 0.15$ GeV for τ near -1, peak moves to lower values $Q_1 = Q_2 = 0.04$ GeV for τ near 1.
Slopes along the two axis and along the diagonal (at $Q_1 = Q_2 = 0$) vanish.
Even for constant form factor, one obtains finite result: $\left(\frac{\alpha}{\pi}\right)^3 a_{\mu;\mathrm{WZW}}^{\mathrm{HLbL};\pi^0(2)} \sim 2.5 \times 10^{-11}$

Pole contributions from η and η'

Only dependence on pseudoscalars appears in weight functions through pseudoscalar mass m_P in propagators:

$$\text{In weight function } w_1(Q_1, Q_1, \tau) : \frac{1}{Q_2^2 + m_P^2}$$

$$\text{In weight function } w_2(Q_1, Q_1, \tau) : \frac{1}{(Q_1 + Q_2)^2 + m_P^2} = \frac{1}{Q_1^2 + 2Q_1 Q_2 \tau + Q_2^2 + m_P^2}$$

Two effects:

1. Shifts the relevant momentum regions (peaks, ridges) to higher momenta for η compared to π^0 and even higher for η'.

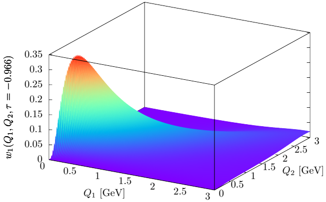
2. Leads to suppression in absolute size of the weight functions due to larger masses in the propagators. For the bulk of the weight functions we have the approximate relations (at same θ, not necessarily at same momenta):

$$w_1\big|_{\eta} \approx \frac{1}{6}\, w_1\big|_{\pi^0}$$

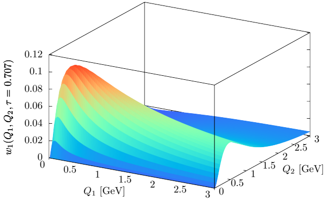
$$w_1\big|_{\eta'} \approx \frac{1}{2.5}\, w_1\big|_{\eta}$$

Weight functions for η

Weight function $w_1(Q_1, Q_2, \tau)$

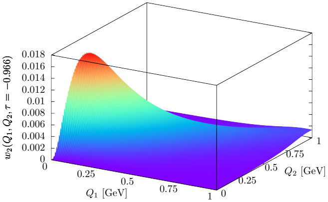


$\tau = -0.966, \quad \theta = 165°$

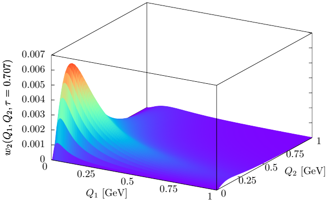


$\tau = 0.707, \quad \theta = 45°$

Weight function $w_2(Q_1, Q_2, \tau)$



$\tau = -0.966, \quad \theta = 165°$

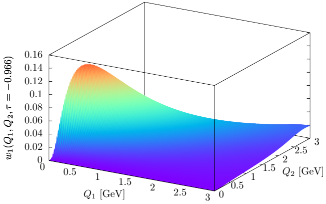


$\tau = 0.707, \quad \theta = 45°$

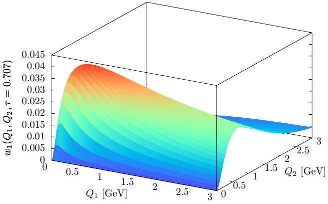
Peaks and ridges broadened compared to π^0.
Peak for w_1 around $Q_1 \sim 0.32 - 0.37$ GeV, $Q_2 \sim 0.22 - 0.33$ GeV.
w_2 about a factor 20 smaller than w_1. Peak for w_2 around $Q_1 = Q_2 \sim 0.14$ GeV for τ near -1, moves down to $Q_1 = Q_2 = 0.06$ GeV for τ near 1.
w_2: finite result for constant form factor $\left(\frac{\alpha}{\pi}\right)^3 a_{\mu;\mathrm{WZW}}^{\mathrm{HLbL};\eta(2)} = 0.78 \times 10^{-11}$

Weight functions for η'

Weight function $w_1(Q_1, Q_2, \tau)$

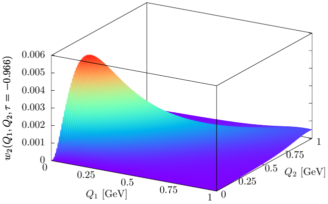


$\tau = -0.966, \quad \theta = 165^\circ$

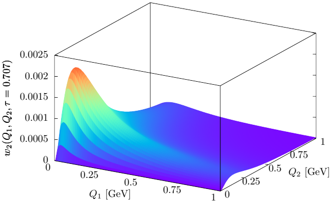


$\tau = 0.707, \quad \theta = 45^\circ$

Weight function $w_2(Q_1, Q_2, \tau)$



$\tau = -0.966, \quad \theta = 165^\circ$



$\tau = 0.707, \quad \theta = 45^\circ$

Peaks and ridges have broadened even more compared to η.

Peak for w_1 around $Q_1 \sim 0.41 - 0.51$ GeV, $Q_2 \sim 0.31 - 0.43$ GeV.

w_2 about a factor 20 smaller than w_1. Peak for w_2 around $Q_1 = Q_2 \sim 0.14$ GeV for τ near -1, moves down to $Q_1 = Q_2 = 0.07$ GeV for τ near 1.

w_2: finite result for constant form factor $\left(\frac{\alpha}{\pi}\right)^3 a_{\mu;\mathrm{WZW}}^{\mathrm{HLbL};\eta'(2)} = 0.65 \times 10^{-11}$

Relevant momentum regions in $a_\mu^{\mathrm{HLbL;P}}$

For illustration: LMD+V and VMD models

- Since integral $a_\mu^{\mathrm{HLbL};\pi^0(1)}$ is divergent without form factors, we take two simple models for illustration to see where are the relevant momentum regions in the integral.
- Of course, in the end, the models have to be replaced by experimental data on the doubly-virtual form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$.
- LMD+V model (Lowest Meson Dominance + V) is generalization of Vector Meson Dominance (VMD) in framework of large-N_C QCD, which respects (some) short-distance constraints from operator product expansion (OPE).
- Main difference is doubly-virtual case: VMD model violates OPE, falls off too fast:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\mathrm{VMD}}(-Q^2, -Q^2) \sim \frac{1}{Q^4} \quad \text{for large } Q^2$$

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\mathrm{LMD+V}}(-Q^2, -Q^2) \sim \mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\mathrm{OPE}}(-Q^2, -Q^2) \sim \frac{1}{Q^2} \quad \text{for large } Q^2$$

- η, η': use VMD model with adapted parameter F_{P} to describe $\Gamma(\mathrm{P} \to \gamma\gamma)$ and M_V from fit of $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q^2, 0)$ to CLEO data.

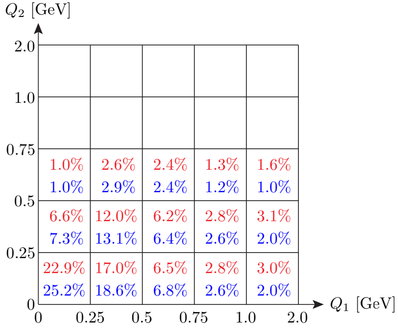
Contributions to $a_\mu^{\mathrm{HLbL};\pi^0}$ from different momentum regions

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = 62.9 \times 10^{-11}$$
$$a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\pi^0} = 57.0 \times 10^{-11}$$

Integrate over momentum bins:

$$\int_{Q_{1,\min}}^{Q_{1,\max}} dQ_1 \int_{Q_{2,\min}}^{Q_{2,\max}} dQ_2 \int_{-1}^{1} d\tau$$

Contribution of individual bins to total:



Bin sizes vary. No entry: contribution $< 1\%$.
Asymmetry in (Q_1, Q_2)-plane with larger contributions below diagonal reflects ridge-like structure in dominant $w_1(Q_1, Q_2, \tau)$.

Pion-pole contribution $a_\mu^{\mathrm{HLbL};\pi^0} \times 10^{11}$ for LMD+V and VMD form factors obtained with momentum cutoff Λ.

Λ [GeV]	LMD+V	VMD
0.25	14.4 (22.9%)	14.4 (25.2%)
0.5	36.8 (58.5%)	36.6 (64.2%)
0.75	48.5 (77.1%)	47.7 (83.8%)
1.0	54.1 (86.0%)	52.6 (92.3%)
1.5	58.8 (93.4%)	55.8 (97.8%)
2.0	60.5 (96.2%)	56.5 (99.2%)
5.0	62.5 (99.4%)	56.9 (99.9%)
20.0	62.9 (100%)	57.0 (100%)

LMD+V and VMD: almost identical absolute contributions below $\Lambda = 0.5$ GeV (0.75 GeV), form factors differ by less than 3% (10%).

Region below $\Lambda = 0.5$ GeV gives more than half of the contribution: 59% for LMD+V, 64% for VMD.

Bulk of result below $\Lambda = 1$ GeV: 86% for LMD+V, 92% for VMD.

VMD: faster fall-off at high momenta $\Rightarrow$ overall smaller contribution compared to LMD+V.

Contributions to $a_\mu^{\mathrm{HLbL};\eta}$ and $a_\mu^{\mathrm{HLbL};\eta'}$ from different momentum regions

One obtains with VMD model:

$$a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\eta} = 14.5 \times 10^{-11}$$

$$a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\eta'} = 12.5 \times 10^{-11}$$

Contribution of individual bins to total (bin sizes vary; no entry: contribution $< 1\%$):

Q_2 [GeV] \ Q_1 [GeV]	0 – 0.25	0.25 – 0.5	0.5 – 0.75	0.75 – 1.0	1.0 – 2.0
1.0 – 2.0		 1.4%	1.0% 1.9%	 1.8%	1.6% 3.3%
0.75 – 1.0		1.7% 2.5%	1.9% 2.9%	1.4% 2.2%	1.5% 2.5%
0.5 – 0.75	2.3% 2.5%	6.2% 7.0%	5.3% 6.4%	2.7% 3.5%	2.3% 3.2%
0.25 – 0.5	8.8% 7.1%	16.6% 14.1%	8.5% 7.8%	3.5% 3.4%	2.7% 2.9%
0 – 0.25	12.1% 7.9%	10.0% 7.0%	3.7% 2.8%	1.4% 1.2%	1.1%

Pole contributions $a_\mu^{\mathrm{HLbL};\eta} \times 10^{11}$ and $a_\mu^{\mathrm{HLbL};\eta'} \times 10^{11}$ with VMD form factor obtained with momentum cutoff Λ.

Λ [GeV]	η	η'
0.25	1.8 (12.1%)	1.0 (7.9%)
0.5	6.9 (47.5%)	4.5 (36.1%)
0.75	10.7 (73.4%)	7.8 (62.5%)
1.0	12.6 (86.6%)	9.9 (79.1%)
1.5	14.0 (96.1%)	11.7 (93.1%)
2.0	14.3 (98.6%)	12.2 (97.4%)
5.0	14.5 (100%)	12.5 (99.9%)
20.0	14.5 (100%)	12.5 (100%)

Region below $\Lambda = 0.25$ GeV gives very small contribution to total: 12% for η, 8% for η'.

Region below $\Lambda = 0.5$ GeV gives: 48% for η, 36% for η'.

Bulk of result below $\Lambda = 1.5$ GeV: 96% for η, 93% for η'.

VMD model might underestimate contribution due to too fast fall-off.

Parametrization of form factor uncertainties

Single-virtual form factor: rough description of measurement errors

$$\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q^2,0) \to \mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q^2,0)\,(1+\delta_{1,\mathrm{P}}(Q))$$

where we assume the following momentum dependent errors:

Region [GeV]	$\delta_{1,\pi^0}(Q)$	$\delta_{1,\eta}(Q)$	$\delta_{1,\eta'}(Q)$
$0 \leq Q < 0.5$	5% [2%]	10%	6%
$0.5 \leq Q < 1$	7% [4%]	15%	11%
$1 \leq Q < 2$	8%	8%	7%
$2 \leq Q$	4%	4%	4%

Error estimates based on:

π^0: $\Gamma(\pi^0 \to \gamma\gamma)$ from PrimEx; TFF in spacelike region from CELLO, CLEO, BABAR, Belle, ongoing analysis by BESIII (future KLOE-2 ?)
η: $\Gamma(\eta \to \gamma\gamma)$ from KLOE-2; spacelike TFF: in addition TPC/2γ; timelike TFF from single Dalitz decays $\eta \to \ell^+\ell^-\gamma$ (NA60: $\ell = \mu$; A2: $\ell = e$)
η': $\Gamma(\eta' \to \gamma\gamma)$ from L3; spacelike TFF below 0.5 GeV from L3 (untagged); timelike TFF from $\eta \to e^+e^-\gamma$ from BESIII
π^0, η: assumed error in lowest momentum bin (no reliable data in spacelike region)
[ ]: use DR for spacelike TFF at low energies (Hoferichter et al. '14)

Parametrization of form factor uncertainties (continued)

Double-virtual form factor: description of measurement errors

$$\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \to \mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\ (1 + \delta_{2,\mathrm{P}}(Q_1, Q_2))$$

$\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$: no experimental data yet.

Measurement planned at BESIII.

Estimate error with Monte Carlo simulations by Mainz group (Denig, Redmer, Wasser) for $e^+e^- \to e^+e^-\gamma^*\gamma^* \to e^+e^-\mathrm{P}$ at BESIII (signal process only !) with EKHARA (Czyż, Ivashyn '11; Czyż et al. '12).

LMD+V model for π^0, VMD model for η, η'.

Uncertainties of double-virtual form factor from Monte Carlo

$\delta_{2,\pi^0}(Q_1, Q_2)$

Q_2 [GeV] \ Q_1 [GeV]	0 – 0.5	0.5 – 0.75	0.75 – 1.0	1.0 – 2.0
1.0 – 2.0	9% (31)	8% (42)	10% (23)	11% (20)
0.75 – 1.0	8% (42)	6% (62)	8% (35)	10% (23)
0.5 – 0.75	9% (32)	5% (86)	6% (62)	8% (42)
0 – 0.5	8%	9% (32)	8% (42)	9% (31)

Note unequal bin sizes !
In brackets: number of MC events N_i in each bin $\sim \sigma \sim \mathcal{F}^2_{\pi^0\gamma^*\gamma^*} \Rightarrow \delta\mathcal{F}_{\pi^0\gamma^*\gamma^*} = \sqrt{N_i}/(2N_i)$ (total: 600 events).
Lowest bin: assumed error.
"Extrapolation" from boundary values (average of neighboring bins), no events in simulation (detector acceptance).

$\delta_{2,\eta}(Q_1, Q_2), \delta_{2,\eta'}(Q_1, Q_2)$

Q_2 [GeV] \ Q_1 [GeV]	0 – 0.5	0.5 – 0.75	0.75 – 1.0	1.0 – 2.0
1.0 – 2.0	11% / 7%	12% / 7%	18% / 10%	25% / 13%
0.75 – 1.0	9% / 6%	9% / 6%	13% / 8%	18% / 10%
0.5 – 0.75	10% / 6%	7% / 4%	9% / 6%	12% / 7%
0 – 0.5	9% / 5%	10% / 6%	9% / 6%	11% / 7%

Top line in bin: η-meson (345 events).
Bottom line: η'-meson (902 events).
Lowest bin: assumed error.

Number of events and corresponding precision for $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ should be achievable with current data set at BESIII plus a few more years of data taking. Separation of signal and background will be more difficult for η and η' than for π^0.

Impact of form factor uncertainties on $a_\mu^{\mathrm{HLbL;P}}$

With the given errors $\delta_{1,\mathrm{P}}(Q)$ and $\delta_{2,\mathrm{P}}(Q_1, Q_2)$ we obtain:

$$
\begin{aligned}
a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} &= 62.9^{+8.9}_{-8.2} \times 10^{-11} \quad \begin{pmatrix} +14.1\% \\ -13.1\% \end{pmatrix} \\
a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\pi^0} &= 57.0^{+7.8}_{-7.3} \times 10^{-11} \quad \begin{pmatrix} +13.7\% \\ -12.7\% \end{pmatrix} \\
a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\eta} &= 14.5^{+3.4}_{-3.0} \times 10^{-11} \quad \begin{pmatrix} +23.4\% \\ -20.8\% \end{pmatrix} \\
a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\eta'} &= 12.5^{+1.9}_{-1.7} \times 10^{-11} \quad \begin{pmatrix} +15.1\% \\ -13.9\% \end{pmatrix}
\end{aligned}
$$

π^0: LMD+V and VMD model yield very similar relative errors. Assume that observations depend little on the used models.

Recall model calculations:

$$
\begin{aligned}
a_\mu^{\mathrm{HLbL};\pi^0} &= (50 - 80) \times 10^{-11} = (65 \pm 15) \times 10^{-11} \ (\pm 23\%) \\
a_\mu^{\mathrm{HLbL;P}} &= (59 - 114) \times 10^{-11} = (87 \pm 27) \times 10^{-11} \ (\pm 31\%)
\end{aligned}
$$

Summary and outlook on pseudoscalar-pole contribution

- Relevant momentum regions in $a_\mu^{\mathrm{HLbL;P}}$ from model-independent weight functions $w_{1,2}(Q_1, Q_2, \tau)$:
  - π^0: < 1 GeV
  - η and η': < 1.5 GeV

- Impact of measurement errors at BESIII of doubly-virtual form factor $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ on $a_\mu^{\mathrm{HLbL;P}}$ based on Monte Carlo simulations for

$$e^+e^- \to e^+e^-\gamma^*\gamma^* \to e^+e^-\mathrm{P}$$

  (reachable in a few more years of data taking and with other assumed input on TFF's at $Q, Q_{1,2} \leq 0.5$ GeV):

$$\begin{aligned}
\delta a_\mu^{\mathrm{HLbL};\pi^0} / a_\mu^{\mathrm{HLbL};\pi^0} &= 14\% \ \ [11\%] \\
\delta a_\mu^{\mathrm{HLbL};\eta} / a_\mu^{\mathrm{HLbL};\eta} &= 23\% \\
\delta a_\mu^{\mathrm{HLbL};\eta'} / a_\mu^{\mathrm{HLbL};\eta'} &= 15\%
\end{aligned}$$

  [ ]: with dispersion relation (DR) for single-virtual $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, 0)$

- In order for dispersive approach to HLbL to be successful, one needs PS-pole contributions to 10% precision $\Rightarrow$ needs to improve uncertainties !

- Future: more work needed to estimate effect of backgrounds and analysis cuts at BESIII. Further informations needed for form factors $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$, in particular for low $Q_{1,2} \leq 1$ GeV, from other experiments (KLOE-2 ? Belle 2 ? CMD-3 ? SND ? Others ?), from DR for form factors and from Lattice QCD.

Muon $g-2$: current status

Contribution	$a_\mu \times 10^{11}$	Reference
QED (leptons)	$116\,584\,718.853 \pm 0.036$	Aoyama et al. '12
Electroweak	153.6 ± 1.0	Gnendiger et al. '13
HVP: LO	6889.1 ± 35.2	Jegerlehner '15
NLO	-99.2 ± 1.0	Jegerlehner '15
NNLO	12.4 ± 0.1	Kurz et al. '14
HLbL	116 ± 40	Jegerlehner, AN '09
NLO	3 ± 2	Colangelo et al. '14
Theory (SM)	$116\,591\,794 \pm 53$	
Experiment	$116\,592\,089 \pm 63$	Bennett et al. '06
Experiment - Theory	295 ± 82	$3.6\ \sigma$

HVP: Hadronic vacuum polarization

HLbL: Hadronic light-by-light scattering

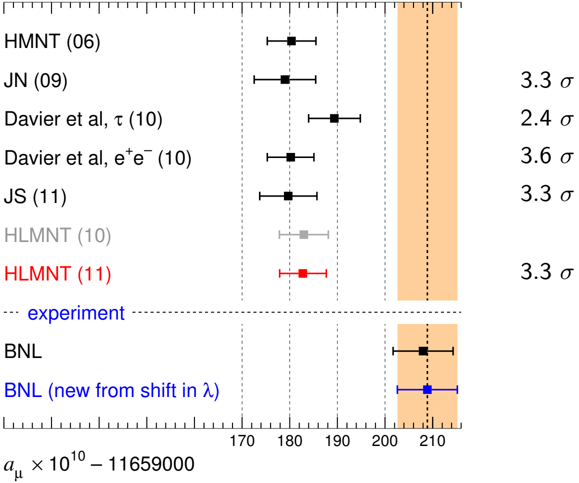
Other estimate: $a_\mu^{\text{HLbL}} = (105 \pm 26) \times 10^{-11}$ (Prades, de Rafael, Vainshtein '09).

Discrepancy a sign of New Physics ?

Hadronic uncertainties need to be better controlled in order to fully profit from future $g-2$ experiments at Fermilab and J-PARC with $\delta a_\mu = 16 \times 10^{-11}$.

Way forward for HVP seems clear: more precise measurements for $\sigma(e^+e^- \to \text{hadrons})$. Not so obvious how to improve HLbL.

Muon $g-2$: other recent evaluations



Source: Hagiwara et al. '11. Note units of 10^{-10} !

Aoyama et al. '12: $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (249 \pm 87) \times 10^{-11}$ [2.9 σ]

Benayoun et al. '15: $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (376.8 \pm 75.3) \times 10^{-11}$ [5.0 σ]

Jegerlehner '15: $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (310 \pm 82) \times 10^{-11}$ [3.8 σ]

New Physics contributions to the muon $g - 2$

Tests of the Standard Model and search for New Physics

- Standard Model (SM) of particle physics very successful in precise description of a huge amount of experimental data, with a few exceptions ($3-4$ standard deviations).

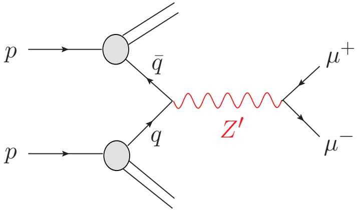
- Some experimental facts (neutrino masses, baryon asymmetry in the universe, dark matter) and some theoretical arguments, which point to New Physics beyond the Standard Model.

- There are several indications that new particles (forces) should show up in the mass range $100 \text{ GeV} - 1 \text{ TeV}$.

Tests of the Standard Model and search for New Physics (continued)

Search for New Physics with two complementary approaches:

1. High Energy Physics:
   e.g. Large Hadron Collider (LHC) at CERN
   Direct production of new particles
   e.g. heavy Z' $\Rightarrow$ resonance peak in invariant mass distribution of $\mu^+\mu^-$ at $M_{Z'}$.

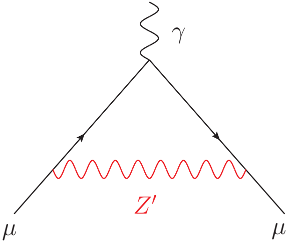


2. Precision physics:
   e.g. anomalous magnetic moments a_e, a_μ
   Indirect effects of virtual particles in quantum corrections
   $\Rightarrow$ Deviations from precise predictions in SM

$$\text{For } M_{Z'} \gg m_\ell : \quad a_\ell \sim \left(\frac{m_\ell}{M_{Z'}}\right)^2$$

Note: there are also non-decoupling contributions of heavy New Physics !
Another example: new light vector meson ("dark photon") with $M_{\gamma'} \sim (10 - 100)$ MeV.

a_e, a_μ allow to exclude some models of New Physics or to constrain their parameter space.



New Physics contributions to the muon $g-2$

Define:

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (290 \pm 90) \times 10^{-11} \qquad \text{(Jegerlehner, AN '09)}$$

Absolute size of discrepancy is actually unexpectedly large, compared to weak contribution (although there is some cancellation there):

$$\begin{aligned} a_\mu^{\text{weak}} &= a_\mu^{\text{weak, (1)}}(W) + a_\mu^{\text{weak, (1)}}(Z) + a_\mu^{\text{weak, (2)}} \\ &= (389 - 194 - 41) \times 10^{-11} \\ &= 154 \times 10^{-11} \end{aligned}$$

Assume that New Physics contribution with $M_{\text{NP}} \gg m_\mu$ decouples:

$$a_\mu^{\text{NP}} = \mathcal{C} \frac{m_\mu^2}{M_{\text{NP}}^2}$$

where naturally $\mathcal{C} = \frac{\alpha}{\pi}$, like from a one-loop QED diagram, but with new particles. Typical New Physics scales required to satisfy $a_\mu^{\text{NP}} = \Delta a_\mu$:

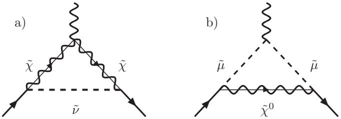
$\mathcal{C}$	1	$\frac{\alpha}{\pi}$	$\left(\frac{\alpha}{\pi}\right)^2$
M_{NP}	$2.0^{+0.4}_{-0.3}$ TeV	100^{+21}_{-13} GeV	5^{+1}_{-1} GeV

Therefore, for New Physics model with particles in $250 - 300$ GeV mass range and electroweak-size couplings $\mathcal{O}(\alpha)$, we need some additional enhancement factor, like large $\tan\beta$ in the MSSM, to explain the discrepancy Δa_μ.

a_μ: Supersymmetry

Supersymmetry for large $\tan\beta, \mu > 0$:

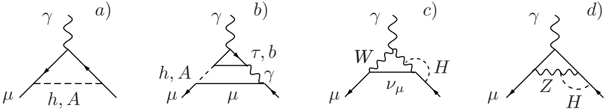
$$a_\mu^{\text{SUSY}} \approx 123 \times 10^{-11} \left(\frac{100 \text{ GeV}}{M_{\text{SUSY}}} \right)^2 \tan\beta$$



(Czarnecki, Marciano, 2001)

Explains $\Delta a_\mu = 290 \times 10^{-11}$ if $M_{\text{SUSY}} \approx (93 - 414)$ GeV $(2 < \tan\beta < 40)$.

In some regions of parameter space, large 2-loop contributions (2HDM):



Barr-Zee diagram (b) yields enhanced contribution, which can exceed 1-loop result. Enhancement factor m_b^2/m_μ^2 compensates suppression by α/π $((\alpha/\pi) \times (m_b^2/m_\mu^2) \sim 4 > 1)$.

a_μ and Supersymmetry after first LHC run

- LHC so far only sensitive to strongly interacting supersymmetric particles, like squarks and gluinos (ruled out below about 1 TeV).
- Muon $g-2$ and SUSY searches at LHC only lead to tension in constrained MSSM (CMSSM) or NUHM1 / NUHM2 (non-universal contributions to Higgs masses).
- In general supersymmetric models (e.g. pMSSM10 = phenomenological MSSM with 10 soft SUSY-breaking parameters) with light neutralinos, charginos and sleptons, one can still explain muon $g-2$ discrepancy and evade bounds from LHC.

a_e, a_μ: Dark photon

In some dark matter scenarios, there is a relatively light, but massive "dark photon" A'_μ that couples to the SM through mixing with the photon:

$$\mathcal{L}_{\rm mix} = \frac{\varepsilon}{2} F^{\mu\nu} F'_{\mu\nu}$$

$\Rightarrow$ A'_μ couples to ordinary charged particles with strength $\varepsilon \cdot e$.
$\Rightarrow$ additional contribution of dark photon with mass $m_{\gamma'}$ to the $g-2$ of a lepton (electron, muon) (Pospelov '09):

$$\begin{aligned} a_\ell^{\text{dark photon}} &= \frac{\alpha}{2\pi}\varepsilon^2 \int_0^1 dx \frac{2x(1-x)^2}{\left[(1-x)^2 + \frac{m_{\gamma'}^2}{m_\ell^2} x\right]} \\ &= \frac{\alpha}{2\pi}\varepsilon^2 \times \begin{cases} 1 & \text{for } m_\ell \gg m_{\gamma'} \\ \frac{2m_\ell^2}{3m_{\gamma'}^2} & \text{for } m_\ell \ll m_{\gamma'} \end{cases} \end{aligned}$$

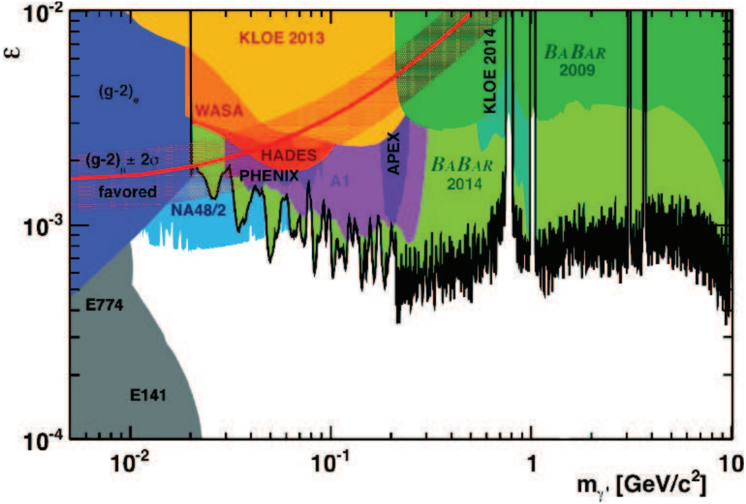
For values $\varepsilon \sim (1-2) \times 10^{-3}$ and $m_{\gamma'} \sim (10-100)$ MeV, the dark photon could explain the discrepancy $\Delta a_\mu = 290 \times 10^{-11}$.

Various searches for the dark photon have been performed, are under way or are planned at BABAR, Jefferson Lab, KLOE, MAMI and other experiments.

For a recent overview, see: *Dark Sectors and New, Light, Weakly-Coupled Particles* (Snowmass 2013), Essig et al., arXiv:1311.0029 [hep-ph].

Status of dark photon searches

Essentially all of the parameter space in the $(m_{\gamma'}, \varepsilon)$-plane to explain the muon $g-2$ discrepancy has now been ruled out.



From: F. Curciarello, FCCP15, Capri, September 2015

Conclusions

- Over many decades, the (anomalous) magnetic moments of the electron and the muon have played a crucial role in atomic and elementary particle physics.
- Experiment and Theory were thereby often going hand-in-hand, pushing each other to the limits.
- From a_e, a_μ we gained important insights into the structure of the fundamental interactions (quantum field theory).
- a_e: Test of QED, precise determination of fine-structure constant α.
  a_μ: Test of Standard Model, potential window to New Physics.

Outlook

- Current situation:

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (295 \pm 82) \times 10^{-11} \qquad [3.6\ \sigma]$$

  Sign of New Physics ? Hadronic effects ? Does $g-2$ experiment measure something different from what is calculated in theory ?

- Two new planned $g-2$ experiments at Fermilab and J-PARC with goal of $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$ (factor 4 improvement)

- Theory needs to match this precision !

- Hadronic vacuum polarization

  Ongoing and planned experiments on $\sigma(e^+e^- \to \text{hadrons})$ with a goal of $\delta a_\mu^{\text{HVP}} = (20-25) \times 10^{-11}$ (factor 2 improvement)

- Hadronic light-by-light scattering

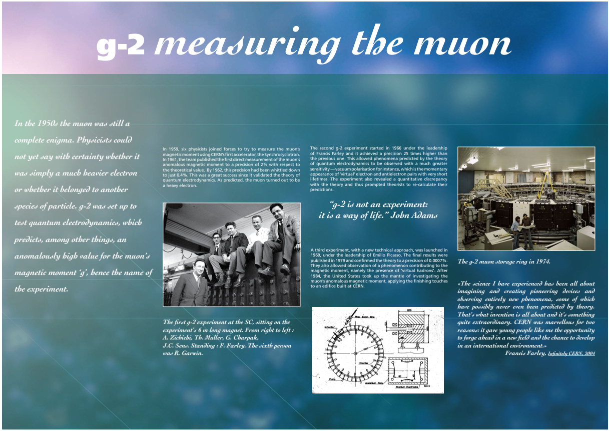
$$a_\mu^{\text{HLbL}} = (105 \pm 26) \times 10^{-11} \qquad \text{(Prades, de Rafael, Vainshtein '09)}$$

$$a_\mu^{\text{HLbL}} = (116 \pm 40) \times 10^{-11} \qquad \text{(AN '09; Jegerlehner, AN '09)}$$

  Error estimates are mostly guesses ! Need a much better understanding of the complicated hadronic dynamics to get reliable error estimate of $\pm 20 \times 10^{-11}$.

  - Better theoretical models needed; more constraints from theory (ChPT, pQCD, OPE); close collaboration of theory and experiment to measure the relevant decays, form factors and cross-sections of hadrons with photons.
  - Promising new data-driven approach using dispersion relations for π^0, η, η' and $\pi\pi$. Still needed: data for scattering of off-shell photons.
  - Future: Lattice QCD.

And finally:



Source: CERN

"$g-2$ is not an experiment: it is a way of life."

John Adams (Head of the Proton Synchrotron at CERN (1954-61) and Director General of CERN (1960-1961))

This statement also applies to many theorists working on the $g-2$!

Backup slides

HLbL scattering: selected results for $a_\mu^{\mathrm{HLbL}} \times 10^{11}$

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 40

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

- Pseudoscalar-exchanges dominate numerically. Other contributions not negligible. Cancellation between π, K-loops and quark loops !
- Note that recent reevaluations of axial vector contribution lead to much smaller estimates than in MV: $a_\mu^{\mathrm{HLbL;axial}} = (8 \pm 3) \times 10^{-11}$ (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). This would shift central values of compilations downwards: $a_\mu^{\mathrm{HLbL}} = (98 \pm 26) \times 10^{-11}$ (PdRV) and $a_\mu^{\mathrm{HLbL}} = (102 \pm 40) \times 10^{-11}$ (N, JN).
- PdRV: Analyzed results obtained by different groups with various models and suggested new estimates for some contributions (shifted central values, enlarged errors). Do not consider dressed light quark loops as separate contribution. Added all errors in quadrature !
- N, JN: New evaluation of pseudoscalar exchange contribution imposing new short-distance constraint on off-shell form factors. Took over most values from BPP, except axial vectors from MV. Added all errors linearly.

Impact of form factor uncertainties on $a_\mu^{\mathrm{HLbL;P}}$: more details

$\frac{\delta a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0}}{a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0}}$	$\frac{\delta a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\pi^0}}{a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\pi^0}}$	$\frac{\delta a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\eta}}{a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\eta}}$	$\frac{\delta a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\eta'}}{a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\eta'}}$	Comment
+14.1% −13.1%	+13.7% −12.7%	+23.4% −20.8%	+15.1% −13.9%	Given δ_1, δ_2
+4.3% −4.2%	+4.4% −4.3%	+6.9% −6.8%	+3.4% −3.3%	Bin $Q < 0.5$ GeV in δ_1 as given, rest: $\delta_{1,2} = 0$
+1.1% −1.0%	+1.0% −0.9%	+4.4% −4.3%	+4.5% −4.4%	Bins $Q \geq 0.5$ GeV in δ_1 as given, rest: $\delta_{1,2} = 0$
+4.5% −4.4%	+4.9% −4.8%	+4.0% −4.0%	+1.7% −1.7%	Bin $Q_{1,2} < 0.5$ GeV in δ_2 as given, rest: $\delta_{1,2} = 0$
+3.9% −3.8%	+3.2% −3.1%	+7.0% −6.8%	+5.1% −5.0%	Bins $Q_{1,2} \geq 0.5$ GeV in δ_2 as given, rest: $\delta_{1,2} = 0$
+10.9% −10.5%	+10.6% −10.1%	–	–	Given δ_1, δ_2, lowest two bins in δ_{1,π^0}: 2%, 4% [DR]
–	–	+20.4% −18.5%	–	Given δ_1, δ_2, lowest two bins in $\delta_{1,\eta}$: 8%, 10%
–	–	–	+13.4% −12.5%	Given δ_1, δ_2, lowest two bins in $\delta_{1,\eta'}$: 5%, 8%
+12.4% −11.6%	+11.8% −11.0%	+22.4% −20.0%	+14.8% −13.6%	π^0, η, η': given δ_1, δ_2, lowest bin δ_2: 5%, 7%, 4%
+12.0% −11.2%	+11.4% −10.6%	+21.9% −19.6%	+14.4% −13.4%	In addition, bins in δ_2 close to lowest: 5%, 7%, 4%

Largest error in red, second largest error in blue.

π^0: for LMD+V FF [VMD FF], region $Q_{1,2} < 0.5$ GeV gives 59% [64%] to total.

For η [η'], region $Q_{1,2} < 0.5$ GeV gives 48% [36%] to total (VMD FF).

$a_\mu^{\mathrm{HLbL};\pi^0}$: to reach goal of 10% error, it would help, if one could measure single- and double-virtual TFF in region $Q, Q_{1,2} < 0.5$ GeV. Assumed error δ_1, δ_2 in lowest bin !

$a_\mu^{\mathrm{HLbL};\eta}, a_\mu^{\mathrm{HLbL};\eta'}$: information for $0.5 \leq Q, Q_{1,2} \leq 1.5$ GeV would be very helpful !

Form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ and transition form factor $F(Q^2)$

- Form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ between an on-shell pion and two off-shell (virtual) photons:

$$i \int d^4x \, e^{iq_1 \cdot x} \langle 0 | T\{j_\mu(x) j_\nu(0)\} | \pi^0(q_1 + q_2) \rangle = \varepsilon_{\mu\nu\alpha\beta} \, q_1^\alpha \, q_2^\beta \, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$

$$j_\mu(x) = (\overline{\psi} \hat{Q} \gamma_\mu \psi)(x), \quad \psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad \hat{Q} = \mathrm{diag}(2, -1, -1)/3$$

(light quark part of electromagnetic current)

Bose symmetry: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2, q_1^2)$

Form factor for real photons is related to $\pi^0 \to \gamma\gamma$ decay width:

$$\mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(q_1^2 = 0, q_2^2 = 0) = \frac{4}{\pi \alpha^2 m_\pi^3} \Gamma_{\pi^0 \to \gamma\gamma}$$

Often normalization with chiral anomaly is used:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, 0) = -\frac{1}{4\pi^2 F_\pi}$$

- Pion-photon transition form factor:

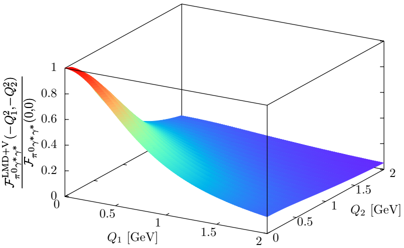
$$F(Q^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, q_2^2 = 0), \qquad Q^2 \equiv -q_1^2$$

Note that $q_2^2 = 0$, but $\vec{q}_2 \neq \vec{0}$ for on-shell photon !

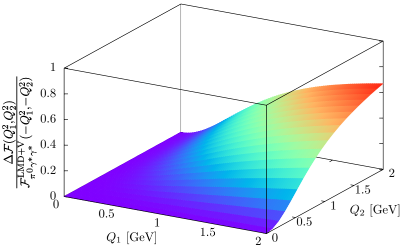
Form factor model: LMD+V (large-N_c QCD) versus VMD

Define: $\Delta\mathcal{F}(Q_1^2, Q_2^2) = \mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) - \mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$

$$\frac{\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)}$$



$$\frac{\Delta\mathcal{F}(Q_1^2,Q_2^2)}{\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}$$



Q_1 [GeV]	Q_2 [GeV]	$\frac{\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)}$	$\frac{\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)}$	$\frac{\Delta\mathcal{F}(Q_1^2,Q_2^2)}{\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)}$
0.5	0	0.707	0.706	0.0003
1	0	0.376	0.376	0.001
0.5	0.5	0.513	0.499	0.027
1	1	0.183	0.141	0.23

For single-virtual FF, both models give equally good fit to CLEO data. Main difference: double-virtual case.

Since LMD+V and VMD FF differ for $Q_1 = Q_2 = 1$ GeV by 23%, it might be possible to distinguish the two models experimentally at BESIII, if binning is chosen properly.

The LMD+V form factor

Knecht, AN, EPJC '01; AN '09

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ in large-N_c QCD in chiral limit with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V).
- $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ fulfills all leading and some subleading QCD short-distance constraints from operator product expansion (OPE).
- Reproduces Brodsky-Lepage (BL): $\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, 0) \sim 1/Q^2$

  (OPE and BL cannot be fulfilled simultaneously with only one vector resonance).
- Normalized to decay width $\Gamma_{\pi^0\to\gamma\gamma}$

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\rm LMD+V}(q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2) + h_1\, (q_1^2 + q_2^2)^2 + \bar{h}_2\, q_1^2\, q_2^2 + \bar{h}_5\, (q_1^2 + q_2^2) + \bar{h}_7}{(q_1^2 - M_{V_1}^2)\, (q_1^2 - M_{V_2}^2)\, (q_2^2 - M_{V_1}^2)\, (q_2^2 - M_{V_2}^2)}$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1465$ MeV

Free model parameters: $h_i, \bar{h}_i$

Transition form factor:

$$F^{\rm LMD+V}(Q^2) = \frac{F_\pi}{3} \frac{1}{M_{V_1}^2 M_{V_2}^2} \frac{h_1 Q^4 - \bar{h}_5 Q^2 + \bar{h}_7}{(Q^2 + M_{V_1}^2)(Q^2 + M_{V_2}^2)}$$

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\rm LMD+V}(-Q^2, 0) \sim 1/Q^2$)
- $\bar{h}_2 = -10.63$ GeV^2 (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $\bar{h}_5 = 6.93 \pm 0.26$ GeV^4 $- h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\rm LMD+V}(-Q^2, 0)$)
- $\bar{h}_7 = -\frac{N_c M_{V_1}^4 M_{V_2}^4}{4\pi^2 F_\pi^2} = -14.83$ GeV^6 (or normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

The VMD form factor

Vector Meson Dominance:

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^{0*}\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_\pi} \frac{M_V^2}{q_1^2 - M_V^2} \frac{M_V^2}{q_2^2 - M_V^2}$$

Only two model parameters: F_π and M_V.

Note:

- VMD form factor factorizes $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = f(q_1^2) \times f(q_2^2)$. This might be a too simplifying assumption / representation.
- VMD form factor has wrong short-distance behavior: $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^4$, for large q^2, falls off too fast compared to OPE prediction $\mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^2$.

Transition form factor:

$$F^{\mathrm{VMD}}(Q^2) = -\frac{N_c}{12\pi^2 F_\pi} \frac{M_V^2}{Q^2 + M_V^2}$$

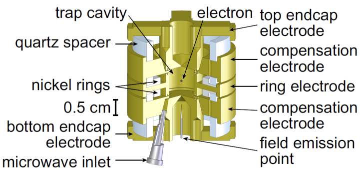
For numerics:

$$\begin{aligned} F_\pi &= 92.4 \text{ MeV}, & M_V = M_\rho = 775.49 \text{ MeV} \\ F_\eta &= 93.0 \text{ MeV}, & M_V = 775 \text{ MeV} \\ F_{\eta'} &= 74.0 \text{ MeV}, & M_V = 859 \text{ MeV} \end{aligned}$$

η, η': F_{P} to describe $\Gamma(\mathrm{P} \to \gamma\gamma)$ and M_V from fit of $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q^2, 0)$ to CLEO data.

Electron $g-2$: Experiment

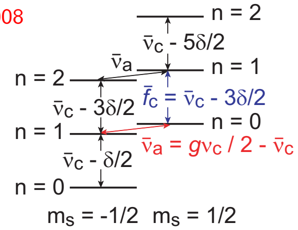
Latest experiment: Hanneke, Fogwell, Gabrielse, 2008



Cylindrical Penning trap for single electron

(1-electron quantum cyclotron)

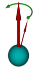
Source: Hanneke et al.



Cyclotron and spin precession levels of electron in Penning trap

Source: Hanneke et al.

$$\frac{g_e}{2} = \frac{\nu_s}{\nu_c} \simeq 1 + \frac{\bar{\nu}_a - \bar{\nu}_z^2/(2\bar{f}_c)}{\bar{f}_c + 3\delta/2 + \bar{\nu}_z^2/(2\bar{f}_c)} + \frac{\Delta g_{cav}}{2}$$



ν_s = spin precession frequency; $\nu_c, \bar{\nu}_c$ = cyclotron frequency: free electron, electron in Penning trap; $\delta/\nu_c = h\nu_c/(m_e c^2) \approx 10^{-9}$ = relativistic correction

4 quantities are measured precisely in experiment:

$\bar{f}_c = \bar{\nu}_c - \frac{3}{2}\delta \approx 149$ GHz; $\bar{\nu}_a = \frac{g}{2}\nu_c - \bar{\nu}_c \approx 173$ MHz;

$\bar{\nu}_z \approx 200$ MHz = oscillation frequency in axial direction;

Δg_{cav} = corrections due to oscillation modes in cavity

$\Rightarrow a_e^{\text{exp}} = 0.00115965218073(28)$ [0.24 ppb ≈ 1 part in 4 billions]

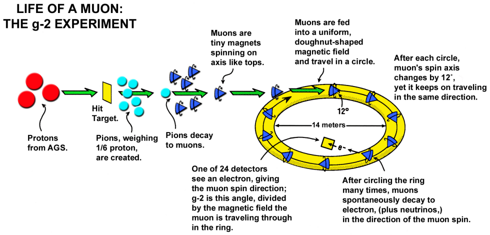
[Kusch & Foley, 1947/48: 4% precision]

Precision in $g_e/2$ even 0.28 ppt ≈ 1 part in 4 trillions !

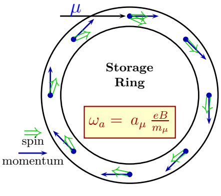
The Brookhaven Muon $g-2$ Experiment

The first measurements of the anomalous magnetic moment of the muon were performed in 1960 at CERN, $a_\mu^{\rm exp} = 0.00113(14)$ (Garwin et al.) [12% precision] and improved until 1979: $a_\mu^{\rm exp} = 0.0011659240(85)$ [7 ppm] (Bailey et al.)

In 1997, a new experiment started at the Brookhaven National Laboratory (BNL):



Source: BNL Muon $g-2$ homepage



actual precession × 2

Angular frequencies for cyclotron precession ω_c and spin precession ω_s:

$$\omega_c = \frac{eB}{m_\mu\,\gamma}\,,\quad \omega_s = \frac{eB}{m_\mu\,\gamma} + a_\mu\,\frac{eB}{m_\mu}\,,\quad \omega_a = a_\mu\,\frac{eB}{m_\mu}$$

$\gamma = 1/\sqrt{1-(v/c)^2}$. With an electric field to focus the muon beam one gets:

$$\vec{\omega}_a = \frac{e}{m_\mu}\left(a_\mu \vec{B} - \left[a_\mu - \frac{1}{\gamma^2-1}\right]\vec{v}\times\vec{E}\right)$$

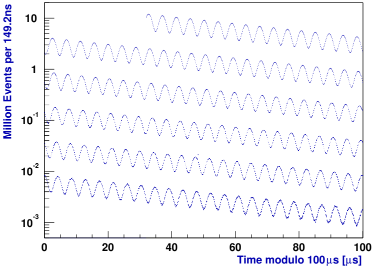
Term with $\vec{E}$ drops out, if $\gamma = \sqrt{1+1/a_\mu} = 29.3$: "magic γ" $\rightarrow p_\mu = 3.094$ GeV$/c$

The Brookhaven Muon $g-2$ Experiment: storage ring

Source: BNL Muon $g-2$ homepage

The Brookhaven Muon $g-2$ Experiment: determination of a_μ

Histogram with 3.6 billion decays of μ^-:



Bennett et al. 2006

$$N(t) = N_0(E)\exp\left(\frac{-t}{\gamma\tau_\mu}\right) \times [1 + A(E)\sin(\omega_a t + \phi(E))]$$

Exponential decay with mean lifetime: $\tau_{\mu,\text{lab}} = \gamma\tau_\mu = 64.378\mu s$ (in lab system).

Oscillations due to angular frequency $\omega_a = a_\mu eB/m_\mu$.

$$a_\mu = \frac{R}{\lambda - R} \text{ where } R = \frac{\omega_a}{\omega_p} \text{ and } \lambda = \frac{\mu_\mu}{\mu_p}$$

Brookhaven experiment measures ω_a and ω_p (spin precession frequency for proton). λ from hyperfine splitting of muonium ($\mu^+ e^-$) (external input).