The muon g-2: a new data-analysis

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Based on
A. Keshavarzi, DN and T. Teubner (KNT)
The \textbf{KNT18} paper is chosen as a Research Highlight in the KEK Annual Report 2018:

4.5 Muon $g-2$ enigma: an omen of new physics?

The anomalous magnetic moment $a_{\mu}$ of the muon, also known as the muon $g - 2$, is one of the most precisely measured quantities in particle physics. In Ref. [1], we calculated the Standard Model (SM) prediction for the muon $g - 2$. Our main conclusion is that there is a discrepancy between the experimental value and SM prediction of the muon $g - 2$ (see Fig. 1). This discrepancy could be a hint of new physics beyond the SM, and hence has drawn the intense attention of many physicists. In view of its importance, Ref. [1] has been selected as an “Editors’ Suggestion” for Physical Review D. Ref. [1] is also featured in “Physics”, which is an online magazine from the American Physical Society to report highlights of papers from the “Physical Review” journals.

Table 1 presents a breakdown of the SM prediction for the muon $g - 2$ together with the current experimental value. The SM prediction can conveniently be separated into three pieces: the QED, electroweak (EW), and hadronic contributions. The QED and EW contributions are perturbatively calculable, and both are known with sufficient precision.

The most problematic contributions are the hadronic contributions. The leading-order (LO) hadronic vacuum polarization (VP) contribution and the hadronic light-by-light (LbL) contribution are the main sources of the uncertainty of the total SM prediction. To obtain a precise SM prediction, it is crucial to evaluate these contributions with sufficient precision.

The VP contributions can be computed more reliably with the help of dispersion relations, using experimental data of $\pi^+ \rightarrow$ hadrons as input. These data have been published in papers, and can be found, e.g., in the Inspire high-energy physics literature database (https://inspirehep.net). The LO hadronic VP contribution $a^{\text{LO VP}}_{\mu}$ can be written as an integral over the total cross-section $a^{\text{LO VP}}(\sigma)$ of the reaction $\pi^+ \rightarrow$ hadrons at the center-of-mass energy $\sqrt{s}$:

$$a^{\text{LO VP}}_{\mu} = \frac{m_{\mu}}{2\pi} \int \frac{d^2 \sqrt{s}}{s^2} \hat{K}(\sigma_{\text{had}}(s)),$$

where $\hat{K}(\sigma)$ is a monotonically increasing function with $\hat{K}(\sigma) \rightarrow 0.40$ and $\hat{K}(\sigma) \rightarrow 1$ for $s \rightarrow \infty$. Because the weight factor in the integrand emphasizes the lowest energy region, good input data at low energies are vital to obtain an accurate prediction for $a^{\text{LO VP}}_{\mu}$. In particular, anonymomestudies 77% of the interest is from state $a^{\pi \pi^*}$, the $e^-e^+ \rightarrow a^{\pi \pi^*}$ data are by far the most important.

To evaluate the integral, we employ data of exclusive measurements over all the hadronic final states for $\sqrt{s} \leq 2$ GeV, and of inclusive measurements at $2 \leq \sqrt{s} \leq 11$ GeV. Above $\sim$11 GeV we employ perturbative QCD.

Our conclusion is that there is a discrepancy between the experimental observation and SM prediction. Some people have attempted to explain the discrepancy in terms of new physics such as axion-like particles. Others have attempted to explain the discrepancy using lattice QCD. There are two ongoing experiments to measure the muon $g - 2$: one at J-PARC and the other at Fermilab. These are aiming to reduce the experimental error by a factor of 10. Once achieved, this experiment will result in further insights into our microscopic world through the muon sector.
Lepton magnetic moment $\bar{\mu}$:

$$\mathcal{H} = -\bar{\mu} \cdot \vec{B}$$

$$\bar{\mu} = -g \frac{e}{2m} \bar{s}, \quad (\bar{s} = \frac{1}{2} \bar{\sigma} \text{ (spin)}, \quad g = 2 + 2F_2(0))$$

where

$$\bar{u}(p + q) \Gamma^\mu u(p) = \bar{u}(p + q) \left( \gamma^\mu F_1(q^2) + \frac{i \sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right) u(p)$$

Anomalous magnetic moment: $a \equiv (g - 2)/2 \ (= F_2(0))$

Historically,

- $g = 2$ (tree level, Dirac)
- $a = \alpha/(2\pi)$ (1-loop QED, Schwinger)

Today, still important, since...

- One of the most precisely measured quantities:

$$a_\mu^{\text{exp}} = 11 659 208.9(6.3) \times 10^{-10} \quad [0.5\text{ppm}] \quad \text{(Bennett et al)}$$

- Extremely useful in probing/constraining physics beyond the SM
Why Muon g-2?

- $\gtrsim 3.5 \sigma$ Anomaly Observed
  Long standing anomaly ($\sim 20$ yrs), in spite of careful studies on every aspect.
  ($\rightarrow$ Major theoretical blunder unlikely.)
  Hint of New Physics beyond the Standard Model?

- No new physics at the LHC so far
  Intensity frontier: more and more important

- Long history of research
  1st $(g - 2)_\mu$ exp.: Garwin, Lederman & Weinrich (1957)
  Well-established place to search for new physics

- Leptonic observable
  Experimentally and theoretically clean
### Muon g-2: previous exp. (after 1960)

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Years</th>
<th>Polarity</th>
<th>$a_\mu \times 10^{10}$</th>
<th>Precision [ppm]</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>CERN I</td>
<td>1961</td>
<td>$\mu^+$</td>
<td>11 450 000 (220 000)</td>
<td>4300</td>
<td>2-loop QED contrib. (3600 ppm)</td>
</tr>
<tr>
<td>CERN II</td>
<td>1962-1968</td>
<td>$\mu^+$</td>
<td>11 661 600 (3100)</td>
<td>270</td>
<td>3-loop QED contrib. (260 ppm)</td>
</tr>
<tr>
<td>CERN III</td>
<td>1974-1976</td>
<td>$\mu^+$</td>
<td>11 659 100 (110)</td>
<td>10</td>
<td>hadronic vacuum polarization contrib. (60 ppm)</td>
</tr>
<tr>
<td>CERN III</td>
<td>1975-1976</td>
<td>$\mu^-$</td>
<td>11 659 360 (120)</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>BNL</td>
<td>1997</td>
<td>$\mu^+$</td>
<td>11 659 251 (150)</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>BNL</td>
<td>1998</td>
<td>$\mu^+$</td>
<td>11 659 191 (59)</td>
<td>5</td>
<td>4-loop QED contrib. (3.3 ppm)</td>
</tr>
<tr>
<td>BNL</td>
<td>1999</td>
<td>$\mu^+$</td>
<td>11 659 202 (15)</td>
<td>1.3</td>
<td>electroweak contrib. (1.3 ppm)</td>
</tr>
<tr>
<td>BNL</td>
<td>2000</td>
<td>$\mu^+$</td>
<td>11 659 204 (9)</td>
<td>0.73</td>
<td>hadronic light-by-light contrib. (0.86 ppm)</td>
</tr>
<tr>
<td>BNL</td>
<td>2001</td>
<td>$\mu^-$</td>
<td>11 659 214 (9)</td>
<td>0.72</td>
<td>hadronic NLO vacuum pol. contrib. (-0.85 ppm)</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td></td>
<td>11 659 208.0 (6.3)</td>
<td>0.54</td>
<td></td>
</tr>
</tbody>
</table>

Table from BNL-E821 final report, Phys. Rev. D 73 (2006) 072003

**History of muon g-2 exp. is a history of SM tests.**

This is not the whole story: the history still goes on.
Basically, any new particle which couples to the muon gives a non-zero contribution to the muon g-2:

- SUSY particles ($\tilde{\mu}, \tilde{W}^\pm, \tilde{Z}^0, \tilde{B}^0, \ldots$)
- extra Higgses ($H^\pm, A^0, H^{\pm\pm}, \ldots$)
- Kaluza-Klein excitations of $\mu$ and $\gamma$
- extra $Z$-like particle ($Z', \text{“dark } Z\text{”}, \ldots$)
- extra $\gamma$-/axion- like light particle ("dark photon", ...)
- leptoquarks

In many cases, the mass and couplings of these new particles are free parameters. By tuning them, one can explain the muon g-2 anomaly. But it is often non-trivial to explain why Nature chooses such a parameter set.
E.g., Family universal type-I 2HDM: Allowed region

Allowed region in the $(\tilde{g}, g')$ plane:

$U(1)_{B-L}$, $z_{\phi_1} = 1$, $\tan \beta = 1$

Just an example when the $U(1)'$ charges = B-L and $\tan \beta = 1$

NA48/2: $\pi^0 \rightarrow Z'\gamma$ searches
E158: Møller scattering
NA64: $e^-$ beam dump exp.
Cs: atomic parity violation in Cs atom

White region is excluded by non-observation of $Z'$ in $^8$Be$^*$' $\rightarrow ^8$Be transition

The strongest constraint comes from atomic parity violation in Cs.

Viable parameter region still exists.

Fig. from L. Delle Rose et al, arXiv:1812.05497
### Breakdown of SM prediction for muon $g-2$

<table>
<thead>
<tr>
<th></th>
<th>2011</th>
<th>2018</th>
<th>2019</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>QED</strong></td>
<td>11658471.81 (0.02)</td>
<td>$\rightarrow$</td>
<td>11658471.90 (0.01)</td>
</tr>
<tr>
<td></td>
<td>[arXiv:1712.06060]</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>EW</strong></td>
<td>15.40 (0.20)</td>
<td>$\rightarrow$</td>
<td>15.36 (0.10)</td>
</tr>
<tr>
<td><strong>LO HLbL</strong></td>
<td>10.50 (2.60)</td>
<td>$\rightarrow$</td>
<td>9.80 (2.60)</td>
</tr>
<tr>
<td></td>
<td>[EPJ Web Conf. 118 (2016)</td>
<td>$\rightarrow$</td>
<td>9.34 (2.92)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>HLMNT11</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>LO HVP</strong></td>
<td>694.91 (4.27)</td>
<td>$\rightarrow$</td>
<td>693.27 (2.46)</td>
</tr>
<tr>
<td><strong>NLO HVP</strong></td>
<td>-9.84 (0.07)</td>
<td>$\rightarrow$</td>
<td>-9.82 (0.04)</td>
</tr>
<tr>
<td><strong>NNLO HVP</strong></td>
<td></td>
<td></td>
<td>1.24 (0.01)</td>
</tr>
<tr>
<td><strong>Theory total</strong></td>
<td>11659182.80 (4.94)</td>
<td>$\rightarrow$</td>
<td>11659182.05 (3.56)</td>
</tr>
<tr>
<td><strong>Experiment</strong></td>
<td></td>
<td></td>
<td>11659209.10 (6.33)</td>
</tr>
<tr>
<td><strong>Exp - Theory</strong></td>
<td>26.1 (8.0)</td>
<td>$\rightarrow$</td>
<td>27.1 (7.3)</td>
</tr>
<tr>
<td><strong>$\Delta a_{\mu}$</strong></td>
<td>3.3$\sigma$</td>
<td>$\rightarrow$</td>
<td>3.7$\sigma$</td>
</tr>
<tr>
<td></td>
<td>(Numbers taken from KNT18 and from KNT19)</td>
<td></td>
<td>3.8$\sigma$</td>
</tr>
</tbody>
</table>

(HVP: Hadronic Vacuum Polarization)
(HLbL: Hadronic Light-by-Light)
QED contribution:

\[ a_\mu^{\text{QED}} = \frac{\alpha}{2\pi} + 0.765857425(17) \left( \frac{\alpha}{\pi} \right)^2 + 24.05050996(32) \left( \frac{\alpha}{\pi} \right)^3 \\
+ 130.8796(63) \left( \frac{\alpha}{\pi} \right)^4 + 753.3(1.0) \left( \frac{\alpha}{\pi} \right)^5 + \cdots \\
= 11658471.895(0.008) \times 10^{-10} , \quad \text{(numbers from PDG 2018)} \]

where the uncertainty is dominated by that of \( \alpha \).

- 5-loop calculation! \quad \text{(Aoyama, Hayakawa, Kinoshita & Nio)}
- The 4-loop corrections \( \simeq 38 \times 10^{-10} \simeq \mathcal{O}(a_\mu^{\text{exp}} - a_\mu^{\text{SM}}) \).
- The 5-loop contribution very small \( \simeq 0.5 \times 10^{-10} \ll a_\mu^{\text{exp}} - a_\mu^{\text{SM}} \)
Electroweak (EW) contribution:

\[ a_\mu (\text{EW}) = 19.48 \times 10^{-10} + (-4.12(10) \times 10^{-10}) + \mathcal{O}(10^{-12}) \]

1-loop

\[ = 15.36(10) \times 10^{-10} , \]

(Number taken from PDG 2018)

where the uncertainty mainly comes from quark loops.

1-loop result published by many groups (Bardeen-Gastmans-Lautrup, Altarelli-Cabibbo-Maiani, Jackiw-Weinberg, Bars-Yoshimura, Fujikawa-Lee-Sanda) in 1972, and now a textbook exercise (Peskin & Schroeder’s textbook, Problems 6.3 (Higgs) and 21.1 (W, Z))

2-loop contribution (∼ 1700 diagrams in the ’t Hooft-Feynman gauge) enhanced by \( \ln(m_Z/m_\mu) \) and also by a factor of \( \mathcal{O}(10) \),

\[ a_\mu (\text{EW, 2-loop}) \simeq -10 \left( \frac{\alpha}{\pi} \right) a_\mu (\text{EW, 1-loop}) \left( \ln \frac{m_Z}{m_\mu} + 1 \right) , \]

where the factor of 10 appears since many “order one” diagrams accidentally add up coherently.
There are several hadronic contributions:

**LO**
- Leading Order (or Vacuum Polarization) Hadronic Contribution

**NLO**
- Next-to-Leading Order Hadronic Contribution

**l-by-l**
- Hadronic light-by-light Contribution

**NNLO Hadronic Contributions**

**Hadronic l-by-l NLO Contrib.**
Modern evaluation of l-by-l contribution

(Melnikov & Vainshtein)

1. First, use the large $N_C$ expansion to find that the leading contribution is the pion pole contribution.

2. Choose the momentum-dependence of the $\pi \gamma \gamma$ coupling (form factor) in such a way that it is consistent with a constraint from QCD (OPE) at the momentum region $q_1^2 \sim q_2^2 \gg q_3^2$. Integrate over the loop momenta.

3. Repeat the above for $\eta, \eta', a_1, \ldots$. Basically that’s all for the LO in $1/N_C$.

4. As for NLO in $1/N_C$, it depends on authors which diagram is numerically important.

For example,

$$
a^\text{lbyl}_\mu = \begin{cases} 
(10.5 \pm 2.6) \times 10^{-10} & \text{‘Glasgow consensus’, arXiv:0901.0306} \\
(9.8 \pm 2.6) \times 10^{-10} & \text{‘G.c.’ w/ correction by Nyffeler, PRD94(2016)053006} \\
(10.2 \pm 3.9) \times 10^{-10} & \text{Nyffeler, arXiv:1710.09742}
\end{cases}
$$
HLbL in muon $g - 2$: summary of selected results (model calculations)

\begin{equation}
\begin{array}{ccc}
\frac{\mu^{-}}{\mu^{-}} = \frac{\pi^0}{\eta} & \frac{\rho}{\rho} & \frac{Q}{Q} \\
\mu^{-} & \mu^{-} & \mu^{-} \\
\end{array}
\end{equation}

Exchange of other resonances $(f_0, a_1, f_2 \ldots)$

\text{de Rafael '94:}

\text{Chiral counting: } p^4

\text{NC-counting: } 1

\text{Contribution to } a_\mu \times 10^{11}:

<table>
<thead>
<tr>
<th>Model</th>
<th>Contribution</th>
<th>ud.:</th>
<th>ud.:</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPP</td>
<td>$+83 (32)$</td>
<td>$-19 (13)$</td>
<td>$+85 (13)$</td>
</tr>
<tr>
<td>HKS</td>
<td>$+90 (15)$</td>
<td>$-5 (8)$</td>
<td>$+83 (6)$</td>
</tr>
<tr>
<td>KN</td>
<td>$+80 (40)$</td>
<td></td>
<td>$+83 (12)$</td>
</tr>
<tr>
<td>MV</td>
<td>$+136 (25)$</td>
<td>$0 (10)$</td>
<td>$+114 (10)$</td>
</tr>
<tr>
<td>2007</td>
<td>$+110 (40)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PdRV</td>
<td>$+105 (26)$</td>
<td>$-19 (19)$</td>
<td>$+114 (13)$</td>
</tr>
<tr>
<td>N,JN</td>
<td>$+116 (39)$</td>
<td>$-19 (13)$</td>
<td>$+99 (16)$</td>
</tr>
</tbody>
</table>

\text{ud.} = \text{undressed, i.e. point vertices without form factors}

\text{Pseudoscalars: numerically dominant contribution (according to most models !).}

Recall (in units of $10^{-11}$): $\delta a_\mu (\text{HVP}) \approx 40$; $\delta a_\mu (\text{exp [BNL]}) = 63$; $\delta a_\mu (\text{future exp}) = 16$

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09 (compilation; "Glasgow consensus"); N,JN = AN '09; Jegerlehner, AN '09 (compilation)

Recent reevaluations of axial vector contribution lead to much smaller estimates than in MV '04:

\begin{equation}
a_{\mu}^{\text{HLbL};\text{axial}} = (8 \pm 3) \times 10^{-11} \quad \text{(Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). Would shift central values of compilations downwards:}
\end{equation}

\begin{equation}
a_{\mu}^{\text{HLbL}} = (98 \pm 26) \times 10^{-11} \quad \text{(PdRV) and} \quad a_{\mu}^{\text{HLbL}} = (102 \pm 39) \times 10^{-11} \quad \text{(N, JN).}
\end{equation}
The diagram to be evaluated:

\[ a^{\text{had,LO}}_{\mu} = \frac{m^2_{\mu}}{12\pi^3} \int_{s_{\text{th}}}^{\infty} ds \frac{1}{s} \hat{K}(s) \sigma_{\text{had}}(s) \]

pQCD not useful. Use the dispersion relation and the optical theorem.

\[ 2 \text{Im} \begin{array}{c} \blacksquare \\square \end{array} = \sum_{\text{had.}} \int d\Phi \left| \begin{array}{c} \blacksquare \\square \end{array} \right|^2 \]

- Weight function \( \hat{K}(s)/s = \mathcal{O}(1)/s \)
  \[ \implies \text{Lower energies more important} \]
  \[ \implies \pi^+\pi^- \text{ channel: 73\% of total } a^{\text{had,LO}}_{\mu} \]
Main improvements between HLMNT11 and KNT18/19

- Lots of new input $\sigma(e^+e^- \rightarrow \text{hadrons})$ data
- Improvements in the estimates of uncertainties due to radiative corrections (Vacuum Polarization Radiative Corrections & Final State Radiations)
- Improvements in data-combination method
Main improvements between HLMNT11 and KNT18/19

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Breakdown of contributions to $a_\mu$ (had, LO VP) from various hadronic final states

We have included new data sets from $\sim 30$ papers, in addition to those included in the HLMNT11 analysis

We have included $\sim 30$ hadronic final states

At $2 \lesssim \sqrt{s} \lesssim 11$ GeV, we use inclusively measured data

At higher energies $\gtrsim 11$ GeV, we use pQCD

<table>
<thead>
<tr>
<th>Channel</th>
<th>Energy range [GeV]</th>
<th>$d_\mu^{[\text{had,LOVP}]} \times 10^{-10}$</th>
<th>$\Delta a_\mu^{[\text{had}(M_Z^2)]} \times 10^4$</th>
<th>New data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^0p$</td>
<td>$m_X \leq \sqrt{s} \leq 0.600$</td>
<td>$0.12 \pm 0.01$</td>
<td>$0.00 \pm 0.00$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$X^+p$</td>
<td>$2m_X \leq \sqrt{s} \leq 0.305$</td>
<td>$0.87 \pm 0.02$</td>
<td>$0.01 \pm 0.00$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$X^0\pi^0$</td>
<td>$3m_X \leq \sqrt{s} \leq 0.660$</td>
<td>$0.01 \pm 0.00$</td>
<td>$0.00 \pm 0.00$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$\eta_f$</td>
<td>$m_\eta \leq \sqrt{s} \leq 0.869$</td>
<td>$0.00 \pm 0.00$</td>
<td>$0.00 \pm 0.00$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

Data based channels ($\sqrt{s} \leq 1.937$ GeV)

<table>
<thead>
<tr>
<th>Channel</th>
<th>Energy range [GeV]</th>
<th>$d_\mu^{[\text{had,LOVP}]} \times 10^{-10}$</th>
<th>$\Delta a_\mu^{[\text{had}(M_Z^2)]} \times 10^4$</th>
<th>New data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z^0\pi$</td>
<td>$0.600 \leq \sqrt{s} \leq 1.350$</td>
<td>$4.46 \pm 0.10$</td>
<td>$0.36 \pm 0.01$</td>
<td>$[65]$</td>
</tr>
<tr>
<td>$Z^+\pi^-$</td>
<td>$0.305 \leq \sqrt{s} \leq 1.937$</td>
<td>$202.9 \pm 197$</td>
<td>$34 \pm 26$</td>
<td>$[34, 35]$</td>
</tr>
<tr>
<td>$Z^+\rho^0$</td>
<td>$0.660 \leq \sqrt{s} \leq 1.937$</td>
<td>$47.79 \pm 0.89$</td>
<td>$4.77 \pm 0.08$</td>
<td>$[36]$</td>
</tr>
<tr>
<td>$Z^0\pi^0\pi^0$</td>
<td>$0.613 \leq \sqrt{s} \leq 1.937$</td>
<td>$14.87 \pm 0.20$</td>
<td>$4.02 \pm 0.05$</td>
<td>$[40, 42]$</td>
</tr>
<tr>
<td>$Z^+\pi^-\pi^0$</td>
<td>$0.850 \leq \sqrt{s} \leq 1.937$</td>
<td>$19.39 \pm 0.78$</td>
<td>$5.00 \pm 0.20$</td>
<td>$[44]$</td>
</tr>
<tr>
<td>$(2\pi \pm 2\eta)^{[\text{had}]}$</td>
<td>$0.913 \leq \sqrt{s} \leq 1.937$</td>
<td>$0.09 \pm 0.09$</td>
<td>$0.33 \pm 0.03$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$3\pi^+\pi^-$</td>
<td>$1.313 \leq \sqrt{s} \leq 1.937$</td>
<td>$0.23 \pm 0.01$</td>
<td>$0.09 \pm 0.01$</td>
<td>$[66]$</td>
</tr>
<tr>
<td>$(2\pi \pm 2\eta)^{[\text{had}]}$</td>
<td>$1.322 \leq \sqrt{s} \leq 1.937$</td>
<td>$0.15 \pm 0.17$</td>
<td>$0.51 \pm 0.06$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

At $2 \lesssim \sqrt{s} \lesssim 11$ GeV, we use inclusively measured data

Table from KNT18, Phys. Rev. D97 (2018) 114025
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- Improvements in data-combination method
Vacuum Polarization Corrections to $\sigma(e^+e^- \rightarrow \text{hadrons})$

Optical Theorem:

Experimentally observed cross section:

To evaluate $\alpha_{\mu}^{\text{LO, had}}$, we need to subtract the vacuum polarization (VP) contribution.

It is straightforward to subtract the leptonic part of the VP, but the hadronic part is non-trivial: we need to do this recursively by using hadronic data. (We did this in the KNT18 paper.)
Optical Theorem:

To evaluate $a_{\mu, \text{had}}^{\text{LO}}$, by definition, we use the hadronic cross sections which include all the Final State Radiations (FSR).

In real experiments, people often impose cuts on the final state photons and/or miss photons in the final states. So we have to add back those missed photons, which introduces uncertainties.

In KNT18, we revisited the FSR corrections in the $K^+ K^-$ and $K^0_S K^0_L$ final states, and found smaller FSR uncertainties than our previous papers.
Main improvements between HLMNT11 and KNT18/19

- Lots of new input $\sigma(e^+e^- \to \text{hadrons})$ data
- Improvements in the estimates of uncertainties due to radiative corrections (Vacuum Polarization Radiative Corrections & Final State Radiations)
- Improvements in data-combination method
Data Combination

To evaluate the vacuum polarization contribution, we have to combine lots of experimental data.

To do so, we usually construct a $\chi^2$ function and find the value of $R(s)$ at each bin which minimizes $\chi^2$.

Naively, the $\chi^2$ function defined as

$$
\chi^2(\{\bar{R}_i\}) \equiv \sum_{n=1}^{N_{\text{exp}}} \sum_{i=1}^{N_{\text{bin}}} \sum_{j=1}^{N_{\text{bin}}} (R_i^{(n)} - \bar{R}_i)(V^{-1}_n)_{ij} (R_j^{(n)} - \bar{R}_j) ,
$$

where $V_n$ is the cov. matrix of the $n$-th exp.,

$$
V_{n,ij} = \begin{cases} 
(\delta R_{i,\text{stat}}^{(n)})^2 + (\delta R_{i,\text{sys}}^{(n)})^2 & \text{(for } i = j) \\
(\delta R_{i,\text{sys}}^{(n)})(\delta R_{j,\text{sys}}^{(n)}) & \text{(for } i \neq j) 
\end{cases}
$$

may seem OK, but when there are non-negligible normalization uncertainties in the data, we have to be more careful.
We first consider an observable $x$ whose true value is 1. Suppose that there is an experiment which measures $x$ and whose normalization uncertainty is 10%. Now, assume that this experiment measured $x$ twice:

1st result: $0.9 \pm 0.1_{\text{stat}} \pm 10\%_{\text{syst}}$, 
2nd result: $1.1 \pm 0.1_{\text{stat}} \pm 10\%_{\text{syst}}$.

Taking the systematic errors 0.09 and 0.11, respectively, the covariance matrix and the $\chi^2$ function are

$$(\text{cov.}) = \begin{pmatrix} 0.1^2 + 0.09^2 & 0.09 \cdot 0.11 \\ 0.09 \cdot 0.11 & 0.1^2 + 0.11^2 \end{pmatrix},$$

$$\chi^2 = (x - 0.9 \quad x - 1.1) (\text{cov.})^{-1} \begin{pmatrix} x - 0.9 \\ x - 1.1 \end{pmatrix}.$$ 

$\chi^2$ takes its minimum at $x = 0.98$: Biased downwards!
d’Agostini bias (2): improvement by iterations

What was wrong? In the previous page,

1st result: $0.9 \pm 0.1_{\text{stat}} \pm 10\%_{\text{syst}}$,

2nd result: $1.1 \pm 0.1_{\text{stat}} \pm 10\%_{\text{syst}}$.

we took the syst. errors 0.09 and 0.11, respectively, which made the downward bias. Instead, we should take 10\% of some estimator $\bar{x}$ as the syst. errors. Then,

$$(\text{cov.}) = \begin{pmatrix} 0.1^2 + (0.1\bar{x})^2 & (0.1\bar{x})^2 \\ (0.1\bar{x})^2 & 0.1^2 + (0.1\bar{x})^2 \end{pmatrix},$$

$$\chi^2 = (x - 0.9 \quad x - 1.1) (\text{cov.})^{-1} \begin{pmatrix} x - 0.9 \\ x - 1.1 \end{pmatrix}.$$ 

$\chi^2$ takes its minimum at $x = 1.00$: Unbiased!

In more general cases, we use iterations: we find an estimator for the next round of iteration by $\chi^2$-minimization. 

$$\sigma(e^+e^- \rightarrow \pi^+\pi^-)$$ data

![Graph showing the cross-section of $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$ vs. $\sqrt{s}$ with various data sets and a fit.]

Fig. from KNT19, arXiv:1911.00367
$\sigma(e^+e^- \rightarrow \pi^+\pi^-)$: $\rho-\omega$ interference region

Fig. from KNT19, arXiv:1911.00367
$\sigma(e^+ e^- \rightarrow \pi^+ \pi^-)$: relative differences

$\sigma_\mu^{\pi^+\pi^-}(0.6 \leq \sqrt{s} \leq 0.9 \text{ GeV}) = (369.84 \pm 1.30) \times 10^{-10}$

Global $\chi^2_{min}$/d.o.f=1.26

Fig. from KNT19, arXiv:1911.00367
Contribution to $(g - 2)_\mu$ from $\pi^+\pi^-$ channel

Fit of all $\pi^+\pi^-$ data: $368.84 \pm 1.30$

Direct scan only: $370.77 \pm 2.61$

KLOE combination: $366.88 \pm 2.15$

BaBar (09): $376.71 \pm 2.72$

BESIII (15): $368.15 \pm 4.22$

CLEO-c (17): $376.69 \pm 7.05$

Fig. from KNT19, arXiv:1911.00367
Other notable exclusive channels [KNT18: arXiv:1802.02995, PRD (in press)]

Results from individual channels

\[ \sigma_{e^+e^- \rightarrow \pi^+\pi^-\pi^0} \]

HLMNT11: 47.51 ± 0.99  
KNT18: 47.92 ± 0.89

\[ \sigma_{e^+e^- \rightarrow K^+K^-} \]

HLMNT11: 22.15 ± 0.46  
KNT18: 23.03 ± 0.22

\[ \sigma_{e^+e^- \rightarrow K^0S K^0_L} \]

HLMNT11: 14.65 ± 0.47  
KNT18: 14.87 ± 0.20

\[ \sigma_{e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0} \]

HLMNT11: 20.37 ± 1.26  
KNT18: 19.39 ± 0.78

Inclusive (\( \sqrt{s} > 2 \) GeV)

HLMNT11: 41.40 ± 0.87  
KNT18: 41.27 ± 0.62

Slide by A. Keshavarzi (Liverpool) at ‘Muon g − 2 Workshop’ at Mainz, June 18-22, 2018
Adding up all the channels, pQCD & narrow resonances contributions, we get

\[
\alpha_{\mu, \text{had, LO VP (KNT19)}} = (692.8 \pm 2.4) \times 10^{-10}
\]
\[
\alpha_{\mu, \text{had, NLO VP (KNT19)}} = (-9.83 \pm 0.04) \times 10^{-10}
\]

(KNT18: \((693.3 \pm 2.5) \times 10^{-10}\))

(KNT18: \((-9.82 \pm 0.04) \times 10^{-10}\))
Breakdown of SM prediction for muon g-2

<table>
<thead>
<tr>
<th></th>
<th>2011</th>
<th>2018</th>
<th>2019</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED</td>
<td>11658471.81 (0.02)</td>
<td>11658471.90 (0.01)</td>
<td>[arXiv:1712.06060]</td>
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<tr>
<td>EW</td>
<td>15.40 (0.20)</td>
<td>15.36 (0.10)</td>
<td>[Phys. Rev. D 88 (2013) 053005]</td>
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<tr>
<td>LO HLbL</td>
<td>10.50 (2.60)</td>
<td>9.80 (2.60)</td>
<td>[EPJ Web Conf. 118 (2016) 01016]</td>
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<tr>
<td>NLO HLbL</td>
<td>0.30 (0.20)</td>
<td></td>
<td>[Phys. Lett. B 735 (2014) 90]</td>
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</table>

<table>
<thead>
<tr>
<th></th>
<th>HLMNT11</th>
<th>KNT18</th>
<th>KNT19</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO HVP</td>
<td>694.91 (4.27)</td>
<td>693.27 (2.46)</td>
<td>692.78 (2.42)</td>
</tr>
<tr>
<td>NLO HVP</td>
<td>-9.84 (0.07)</td>
<td>-9.82 (0.04)</td>
<td>-9.83 (0.04)</td>
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<tr>
<td>NNLO HVP</td>
<td></td>
<td>1.24 (0.01)</td>
<td>[Phys. Lett. B 734 (2014) 144]</td>
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</table>

<table>
<thead>
<tr>
<th></th>
<th>Theory total</th>
<th>11659182.80 (4.94)</th>
<th>11659182.05 (3.56) [this work]</th>
<th>181.08 (3.78)</th>
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<tr>
<td>Experiment</td>
<td></td>
<td>11659209.10 (6.33)</td>
<td>world avg</td>
<td></td>
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<tr>
<td>Exp - Theory</td>
<td>26.1 (8.0)</td>
<td>27.1 (7.3) [this work]</td>
<td>28.0 (7.4)</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\Delta a_\mu$</th>
<th>3.3$\sigma$</th>
<th>3.7$\sigma$ [this work]</th>
<th>3.8$\sigma$</th>
</tr>
</thead>
</table>

(HVP: Hadronic Vacuum Polarization)  (Numbers taken from KNT18 and from KNT19)

(HLbL: Hadronic Light-by-Light)
Comparison with Other Work

Contributions from major channels to $a_\mu$(LO,had) for $\sqrt{s} < 1.8\text{GeV}$:

<table>
<thead>
<tr>
<th>channel</th>
<th>KNT18</th>
<th>DHMZ19</th>
<th>diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+\pi^-$</td>
<td>$503.74 \pm 1.96$</td>
<td>$507.80 \pm 3.35$</td>
<td>$-4.06$</td>
</tr>
<tr>
<td>$\pi^+\pi^-\pi^0$</td>
<td>$47.70 \pm 0.89$</td>
<td>$46.20 \pm 1.45$</td>
<td>$1.50$</td>
</tr>
<tr>
<td>$K^+K^-$</td>
<td>$23.00 \pm 0.22$</td>
<td>$23.08 \pm 0.44$</td>
<td>$-0.08$</td>
</tr>
<tr>
<td>$\pi^+\pi^-2\pi^0$</td>
<td>$18.15 \pm 0.74$</td>
<td>$18.01 \pm 0.55$</td>
<td>$0.14$</td>
</tr>
<tr>
<td>$2\pi^+2\pi^-$</td>
<td>$13.99 \pm 0.19$</td>
<td>$13.68 \pm 0.31$</td>
<td>$0.31$</td>
</tr>
<tr>
<td>$K^0_SK^0_L$</td>
<td>$13.04 \pm 0.19$</td>
<td>$12.82 \pm 0.24$</td>
<td>$0.22$</td>
</tr>
<tr>
<td>$\pi^0\gamma$</td>
<td>$4.58 \pm 0.10$</td>
<td>$4.29 \pm 0.10$</td>
<td>$0.29$</td>
</tr>
</tbody>
</table>


Difference in the $\pi^+\pi^-$ channel is mainly from the way to combine the data sets.

KNT18: Global $\chi^2$ minimization
DHMZ19: Takes the average of “all but KLOE” and “all but BaBar” as the mean value, and counts the half of the diff of the two as an additional systematic uncertainty.
Comparison with Lattice Results

Status of HVP determinations

No new physics
KNT 2018
Jegerlehner 2017
DHMZ 2017
DHMZ 2012
HLMNT 2011
RBC/UKQCD 2018
Mainz 2019
FNAL/HPQCD/MILC 2019
HPQCD 2016
ETMC 2013

Lattice + R-ratio
Lattice
R-ratio

$\alpha_{\mu} \times 10^{10}$

Fig. by C. Lehner (BNL), talk at Lattice 2019
Steering Committee:

G. Colangelo (Hadron Theory)  HVP and HLbL
M. Davier  \((e^+e^- \text{ exp. (BaBar)})\)  HVP
S. Eidelman  \((e^+e^- \text{ exp. (CMD-2, CMD-3 \& SND)})\)  HVP
A. El-Khadra (Lattice QCD)  HVP
C. Lehner (Lattice QCD)  HVP and HLbL
T. Mibe (J-PARC g-2 exp.)  HLbL
A. Nyffeler (Hadron Theory)  HVP
B. L. Roberts (Fermilab g-2 exp.)
T. Teubner (Hadron Theory)  HVP

HVP: Hadronic Vacuum Polarization
HLbL: Hadronic Light-by-Light
Muon g-2 Theory Initiative: Goals

- theory support to the Fermilab and J-PARC experiments to maximize their impact
  - need theoretical predictions of the hadronic corrections with reduced and reliably estimated uncertainties

- summarize the theoretical calculations of the hadronic corrections to the muon g-2
  - comparisons of intermediate quantities between the different approaches. For example, lattice vs experiment
  - assess reliability of uncertainty estimates

- combine to provide theory predictions for $a^\text{HVP}_\mu$ and $a^\text{HLbL}_\mu$ and write a report before the Fermilab and J-PARC experiments announce their first results.

slide by A. El-Khadra at PhiPsi17, June 26-29, 2017
1st plenary workshop: near Fermilab, June 2017

Hadronic Vacuum Polarization workshop: KEK, February 2018

Hadronic Light-by-Light workshop: Connecticut, March 2018

2nd plenary workshop: Mainz, June 2018

3rd plenary workshop: Seattle, September 2019

4th plenary workshop: KEK, June 2020

We have discussed a lot about the **White Paper**: In particular, how to come up with a single theory prediction to be compared with the exp. result.
Earliest possible release date for Fermilab g-2 measurement:
15-20 December 2019

Post the WP on arXiv by:
1 Dec. 2019

Deadline for finalizing individual WP chapters:
1 Nov 2019
At this date the Overleaf chapters will be frozen.

Editorial board will release complete WP to authors for feedback on:
15 Nov. 2019
will need to receive feedback from authors within a week

Experimental and theoretical inputs used in WP must be published by:
15 Oct 2019
To make sure to be included in WP discussion, a paper to be posted in arXiv by same date.

Note: The WP will be posted on arXiv in December, even if the Fermilab experiment’s release date is delayed.

slide by A. El-Khadra at the Seattle muon g-2 workshop, September 9-13, 2019
White Paper Outline

- Executive Summary
- Introduction
- Chapter 1: data-driven HVP
- Chapter 2: lattice HVP
- Chapter 3: data-driven HLbL
- Chapter 4: lattice HLbL
- Chapter 5: QED + EW

T. Aoyama, T. Kinoshita, M. Nio
D. Stöckinger, H. Stöckinger-Kim

Summary, Conclusions, and Outlook

slide by A. El-Khadra at the Seattle muon g-2 workshop, September 9-13, 2019
Muong g-2 theory initiative workshop

June 1-5, 2020
KEK, Tsukuba, Japan

D. Nomura (KEK)

Local organizers:
Motoi Endo
Shoji Hashimoto
Toru Iijima
Daisuke Nomura
Tsutomu Mibe
Kazuhito Suzuki

About

The muon g-2 is arguably one of the most important observables in contemporary particle physics. The long-standing anomaly at the level of more than 3 standard deviations between the experimental value

June 1-5, 2020 at KEK
an activity of the Muon g-2 Theory Initiative
Standard Model prediction for \((g - 2)_\mu: \gtrsim 3.5\sigma\) deviation from measured value \(\implies\) New Physics?

Recent data-driven evaluations of hadronic vacuum polarization contributions seem convergent

To better establish the \(g - 2\) anomaly, better data for \(e^+e^- \rightarrow \pi^+\pi^-\) welcome (from CMD-3, SND, Belle II, . . .)

Lattice calculations still suffer from large uncertainties (but a hybrid approach is useful)

New exp. at Fermilab and J-PARC expected to reduce the uncertainty of \((g - 2)_\mu\) by a factor of 4